Classifying Groups with a large Subgroup
A status report

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August 29, 2019

Large subgroups

We fix a prime p.

We will consider groups G with $O_p(G) = 1$, with a so called large subgroup. It is a p-subgroup Q such that

1. $C_G(Q) \leq Q$.
2. If $1 \neq U \leq Z(Q)$, then $N_G(U) \leq N_G(Q)$.

If a group G contains such a group Q, we also speak about Q-uniqueness.
Without loss we may additionally assume that

$$Q = O_p(N_G(Q)).$$

Groups of local/parabolic characteristic p

Definition

Let G be a group, L be a p-local subgroup of G, then L is called of characteristic p if $F^*(L) = O_p(L)$.(or $C_L(O_p(L)) \leq O_p(L)$)
We call G of local characteristic p if all p-local subgroups of G are of characteristic p
We call G of parabolic characteristic p if all p-local subgroups L, which contain a Sylow p-subgroup of G, are of characteristic p.

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Notation

Let $X \subseteq G$, then $\mathcal{L}_G(X) = \{L \leq G, X \subseteq L, O_p(L) \neq 1\}$.

Some notation

Further notation

We fix $S \in \mathrm{Syl}_p(G)$, with $Q \leq S$, $Z = \Omega_1(Z(S))$ and set
$C = N_G(Z)$
Further we denote by $\tilde{C} = N_G(Q)$. Then $C \leq \tilde{C}$.

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Further we denote by $\tilde{C} = N_G(Q)$. Then $C \leq \tilde{C}$.

For $L \in \mathcal{L}_G(S)$ set Y_L, the maximal p-reduced elementary abelian normal subgroup of L, i.e. $O_p(L/C_L(Y_L)) = 1$.

The Local Structure Theorem (U. Meierfrankenfeld, B. Stellmacher, S. 2016)

The structure theorem deals with the p-local subgroups of G, which are different from $\tilde{C}$. More precisely:

For $L \in \mathcal{L}_G(S)$ with $L \not\leq N_G(Q)$, the Local Structure Theorem provides information about $\tilde{L^\circ} = L^\circ / C_{L^\circ}(Y_L)$ and its action on Y_L, where $L^\circ = \langle Q^L \rangle$. Recall $L = L^\circ N_L(Q)$.

Example 1

(2) The symplectic case.

(a) $\tilde{L^\circ} \cong \mathrm{Sp}_{2n}(q), n \geq 2$, or $\mathrm{Sp}_4(2)'$ and $[Y_L, L^\circ]$ is the corresponding natural module for $\tilde{L^\circ}$

(b) If $Y_L \neq [Y_L, L^\circ]$, then $p = 2$ and $|Y_L/[Y_L, L^\circ]| \leq q$.

(c) If $Y_L \not\leq Q$, then $p = 2$ and $[Y_L, L^\circ] \not\leq Q$

(d) If L° is not maximal, then $L^\circ < M^\circ$ and one of the following holds

  (1) $p = 2$, $\tilde{L^\circ} \cong \mathrm{Sp}_4(2)'$, $Y_L = [Y_L, L^\circ] \not\leq Q$, $M^\circ/C_{M^\circ}(Y_M) \cong \mathrm{Mat}(24)$ and Y_M is the simple Golay code module of F_2-dimension 11 for M°.

  (2) $p = 2$ $\tilde{L^\circ} \cong \mathrm{Sp}_4(2)$, $|Y_L/[Y_L, L^\circ]| = 2$, $[Y_L, L^\circ] \not\leq Q$, $M^\circ/C_{M^\circ}(Y_M) \cong \mathrm{Aut}(\mathrm{Mat}(22))$ and Y_M is the simple Tood module of F_2-dimension 10 for M°.

Example 2

(8) (The exceptional case) $Y_L \not\leq Q$ and one of the following holds
   (1) $\tilde{L}^\circ \cong Spin^+_{10}(q)$ and Y_L is the half-spin module
   (2) $\tilde{L}^\circ \cong E_6(q)$, Y_L is one of the two $GF(p)$-modules of order q^{27}.

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(2) $\tilde{L}^\circ \cong E_6(q)$, Y_L is one of the two $GF(p)$-modules of order q^{27}.

(10) There exists some $1 \neq y \in Y_L$ such that $C_G(y)$ is not of characteristic p.

(4) $p = 2$, $\tilde{L} \cong \mathrm{O}_{2n}^\pm(2)$, $\tilde{L^\circ} \cong \Omega_{2n}^\pm(2)$, $2n \geq 4$ and $(2n, \pm) \neq (4, +)$, $[Y_L, L]$ is the corresponding natural module and $Y_L \leq Q$.

The H-structure Theorem (U. Meierfrankenfeld, Chr. Parker, S.), $p = 2$

Theorem

Suppose that G is a finite $\mathcal{K}_p$-group, S a Sylow p-subgroup of G and $Q \leq S$ is a large subgroup of G with $Q = O_p(N_G(Q))$. If there exists $L \in \mathcal{L}_G(S)$ with $Y_L \not\leq Q$, then one of the following holds:

(1) $p = 2$ *and one of the following holds*

- (i) $F^*(G) \cong \mathrm{PSL}_n(q)$, $n \geq 3$, $\mathrm{PSU}_n(q)$, $n \geq 4$, $\Omega^{\pm}_{2n}(q)$, $n \geq 4$, $\mathrm{E}_n(q)$, $n = 6, 7, 8$, *or* ${}^2\mathrm{E}_6(q)$ *where* $q = 2^f$ *and* $f \geq 1$ *is arbitrary; or*
- (ii) $F^*(G) \cong \mathrm{G}_2(3)$, $\mathrm{PSL}_4(3)$, $\mathrm{PSL}_2(9)$, $\mathrm{PSU}_4(3)$, $\mathrm{P}\Omega^{+}_{8}(3)$, $\mathrm{Mat}(22)$, $\mathrm{Mat}(23)$, $\mathrm{Mat}(24)$, He, Suz, Co_1, Co_2, J_4, $\mathrm{M}(22)$, $\mathrm{M}(24)'$, F_2, F_1, $\mathrm{Alt}(9)$, *or* $\mathrm{Alt}(10)$.

The H-structure Theorem, p odd

Theorem

(2) *p is odd and one of the following holds*

(i) $p = 3$ *and* $F^*(G) \cong \mathrm{PSU}_6(2)$, $\Omega_8^+(2)$, $\mathrm{F}_4(2)$, ${}^2\mathrm{E}_6(2)$, McL, $\mathrm{M}(22)$, $\mathrm{M}(24)'$, Co_1, Co_2, Co_3, HN, F_1; *or*

(ii) $F^*(\langle \mathcal{L}_G(S) \rangle)$ *is a simple group of Lie type in characteristic p and of rank at least* 3. *If, in addition, G is a $\mathcal{K}_2$- group of local characteristic p, then* $F^*(\langle \mathcal{L}_G(S) \rangle) = F^*(G) \cong \mathrm{PSL}_n(q)$, $n \geq 4$, $\mathrm{PSU}_n(q)$, $n \geq 6$, $\mathrm{P}\Omega_n^{\pm}(q)$, $n \geq 7$, $\mathrm{PSp}_{2n}(q)$, $n \geq 3$, $\mathrm{E}_6(q)$, $\mathrm{E}_7(q)$, $\mathrm{E}_8(q)$, ${}^2\mathrm{E}_6(q)$ *or* $\mathrm{F}_4(q)$ *where $q = p^f$ and $f \geq 1$ is arbitrary; or*

(iii) *There is a weak BN-pair (P_1, P_2), $S \leq P_1 \cap P_2$, $O_p(\langle P_1, P_2 \rangle) = 1$ such that (P_1, P_2) is of type* $\mathrm{PSL}_3(q)$, $\mathrm{PSU}_4(q)$, $\mathrm{PSU}_5(q)$ *or* $\mathrm{PSp}_4(q)$ *where $q = p^f$ and $f \geq 1$ is arbitrary.*

An example, $G \cong \mathrm{Co}_3$

We consider $p = 3$, $\tilde{L}^\circ \cong \mathrm{Mat}(11)$ and Y_L the 5-dimensional module. We show

$$|Y_L \cap Q| = 3^3 \text{ and } [Q, Y_L \cap Q] = Z(S) = Z.$$

Set $D = \langle Y_L^{\tilde{C}} \rangle (L \cap \tilde{C})$ and $W_D = \langle (Y_L \cap Q)^D \rangle$.

Then $[W_D, W_D] \leq [W_D, Q] = [(Y_L \cap Q), Q]^D = Z$.

Hence W_D is extraspecial. Further $[O_3(L), W_D, W_D] \leq Y_L$, i.e. W_D acts quadratically on $O_3(L)/Y_L$.

As $\mathrm{Mat}(11)$ has no quadratic modules, we get that $[\tilde{L}^\circ, O_3(L)] = Y_L$ and then $O_3(L) = Y_L$. Thus $|S| = 3^7$ and $W_D = Q$.

As Y_L induces a quadratically acting group of order 9 on Q and we are in $\mathrm{Sp}_4(3)$, we see that $O^{3'}(\tilde{C}) = 3^{1+4} SL_2(9)$.

Weak BN-pair

We consider the situation that $\tilde{L}^\circ \cong \mathrm{SL}_2(q^2)$ and $O^{2'}(\tilde{C}/Q) \cong \mathrm{SL}_2(q)$, $q = 2^n > 2$.

Let $B = B_1 B_2$ be a Borel subgroup, where B_1 is one of $L^\circ S$ and B_2 one of $\tilde{C}$ both containing S. Let Γ be the coset graph of the amalgam $(L^\circ B, O^{2'}(\tilde{C})B)$.

Let F be a Cartan subgroup in B and Γ^F be the fixed point graph of F.

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Step 2: z fixes a vertex $\alpha \in \Gamma^F$ opposite to 1.

Weak BN-pair

We show that there is a vertex β of distance two of 1 such that $z \in Z(Q_\beta)$.
Let us assume that α is of type 1. Then we have symmetry. We show that there is a vertex δ of distance two of α with $z \in Z(Q_\delta)$.

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Now we see that $d(1, \alpha) = 4$. We act with F on the path from 1 to α, which gives a second path of length 4 and so an 8-circuit. By arc-transitive action of G, we get that Γ is the incidence graph of a generalized 4-gon.

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Now we see that $d(1, \alpha) = 4$. We act with F on the path from 1 to α, which gives a second path of length 4 and so an 8-circuit. By arc-transitive action of G, we get that Γ is the incidence graph of a generalized 4-gon.

Step 5: Γ is Moufang

By Tits-Weiss we receive $G \cong \mathrm{PSU}_4(q)$.

$Y_L \leq Q$ for all $L \in \mathcal{L}_G(S)$

Theorem (The structure theorem for local characteristic p)

Let G be a $\mathcal{K}_p$-group of local characteristic p and $S \in \mathrm{Syl}_p(G)$. For any $L \in \mathcal{L}_G(S)$ we have that $Y_L \leq Q$. Then for $\tilde{L^\circ}$ one of the following holds:

(1) $F^*(\tilde{L^\circ}) \cong \mathrm{SL}_n(p^m)$, $n \geq 2$, $\mathrm{Sp}_{2n}(p^m)$, $n \geq 2$, or $\mathrm{Sp}_4(2)'$ (and $p = 2$) and $[Y_L, L^\circ]$ is the natural module. Moreover

- (i) $Y_L = [Y_L, L^\circ]$ or $p = 2$ and $\tilde{L^\circ} \cong \mathrm{Sp}_{2n}(q)$, and
- (ii) either $C_{L^\circ}(Y_L) = O_2(L^\circ)$ or $p = 2$ and $L^\circ/O_2(L^\circ) \cong 3\mathrm{Sp}_4(2)'$.

(2) We have the wreath product case.

The wreath product case

There exists a unique $\tilde{L}$-invariant set $\mathcal{K}$, $|\mathcal{K}| > 1$, of subgroups of $\tilde{L}$ such that $Y_L = [Y_L, L^\circ]$ is a natural $SL_2(q)$-wreath product module for $\tilde{L}$ with respect to $\mathcal{K}$.
Moreover $\tilde{L^\circ} = O^p(\langle \mathcal{K} \rangle \tilde{Q}$ and Q acts transitively on $\mathcal{K}$.

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Moreover $\tilde{L^\circ} = O^p(\langle \mathcal{K} \rangle \tilde{Q}$ and Q acts transitively on $\mathcal{K}$.

Natural wreath product module

A natural $SL_2(q)$-wreath product module V with respect to $\mathcal{K}$ is as follows:

$$V = \bigoplus_{K \in \mathcal{K}} [V, K] \text{ and } \langle \mathcal{K} \rangle = \times_{K \in \mathcal{K}} K,$$

and for each $K \in \mathcal{K}$, $K \cong \mathrm{SL}_2(q)$ and $[V, K]$ is the natural $\mathrm{SL}_2(q)$-module for K.

$Y_L \leq Q$ for all $L \in \mathcal{L}_G(S)$, the $P!$-Theorem (Chr. Parker, G. Parmeggiani, B.Stellmacher 2003), $p = 2$

Let $S \leq P$. We call P a minimal parabolic subgroup if S is not normal in P and S is contained in a unique maximal subgroup of P We denote by
$\mathcal{P}_G(S) = \{P \in \mathcal{L}_G(S), P$ a minimal parabolic subgroup$\}$

Theorem

Let G be of local characteristic 2. Assume there is $P \in \mathcal{P}_G(S)$ with $P \not\leq \tilde{C}$. Then

(i) *$P^\circ/O_2(P^\circ) \cong SL_2(2^n)$ and Y_P is the natural module.*

(ii) *P is uniquely determined.*

$Y_L \leq Q$ for all $L \in \mathcal{L}_G(S)$, the $\tilde{P}$!-Theorem (M. Mainardis, U. Meierfrankenfeld, G. Parmeggiani, B. Stellmacher 2005), $p = 2$

Theorem

Let G be of local characteristic 2. Suppose that there exists $P \in \mathcal{P}_G(S)$ such that $P \not\leq \tilde{C}$.

Then there exists at most one $\tilde{P} \in \mathcal{P}_G(S)$ such that $\tilde{P} \not\leq N_G(P^\circ)$ and $\langle P, \tilde{P} \rangle \in \mathcal{L}_G(S)$.

Moreover, if such $\tilde{P}$ exists and $M_1 := \langle P, \tilde{P} \rangle^\circ C_S(Y_P)$, then $M_1/C_{M_1}(Y_{M_1}) \cong SL_3(2^n)$ or $Sp_4(2^n)'$, and $[Y_{M_1}, M_1]$ is the corresponding natural module for $M_1/C_{M_1}(Y_{M_1})$.

$Y_L \leq Q$ for all $L \in \mathcal{L}_G(S)$, Baumann characteristic 2

Theorem (The structure theorem for $Y_L \leq Q$)

Let G be a $\mathcal{K}_p$-group and $S \in \mathrm{Syl}_p(G)$. For any $L \in \mathcal{L}_G(S)$ we have that $Y_L \leq Q$. Then for $\tilde{L^\circ}$ one of the following holds:

(1) *$F^*(\tilde{L^\circ}) \cong \mathrm{SL}_n(p^m)$, $n \geq 2$, $\mathrm{Sp}_{2n}(p^m)'$ and $[Y_L, L^\circ]$ is the natural module. Moreover either $C_{L^\circ}(Y_L) = O_2(L^\circ)$ or $L^\circ/O_2(L^\circ) \cong 3\mathrm{Sp}_4(2)'$.*

(2) *We have the wreath product case.*

(3) *$\tilde{L^\circ} \cong \Omega^{\pm}_{2n}(2)$, $2n \geq 6$, not $\Omega^+_4(2)$, and $[Y_L, L^\circ]$ is the corresponding natural module and $Y_L \leq Q$.*

(4) *Y_L is tall and asymmetric in G, but Y_L is not char p-tall in G.*

tall

Y_L is tall if for $T \in \mathrm{Syl}_p(C_L(Y_L))$ there is a p-local subgroup R, $T \leq R$ with $Y_L \not\leq O_p(R)$

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Y_L is asymmetric if for all $g \in G$ with $[Y_L, Y_L^g] \leq Y_L \cap Y_L^g$ it follows $[Y_L, Y_L^g] = 1$.

$Y_L \leq Q$ for all $L \in \mathcal{L}_G(S)$, Baumann characteristic 2

Theorem (The structure theorem for Baumann characteristic 2)

Let G be a $\mathcal{K}_2$-group of Baumann characteristic 2 and $S \in \mathrm{Syl}_2(G)$. For any $L \in \mathcal{L}_G(S)$ we have that $Y_L \leq Q$. Then for $\tilde{L}^\circ$ one of the following holds:

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$Y_L \leq Q$ for all $L \in \mathcal{L}_G(S)$, Baumann characteristic 2

Theorem (P!-Theorem)

Let G be of Baumann characteristic 2. Assume there is $P \in \mathcal{P}_G(S)$ with $P \not\leq \tilde{C}$. Then

(i) *$P^\circ/O_2(P^\circ) \cong SL_2(2^n)$ and Y_P is the natural module.*

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Theorem ($\tilde{P}$!-Theorem)

Let G be of Baumann characteristic 2. Suppose that there exists $P \in \mathcal{P}_G(S)$ such that $P \not\leq \tilde{C}$. Then there exists at most one $\tilde{P} \in \mathcal{P}_G(S)$ such that $\tilde{P} \not\leq N_G(P^\circ)$ and $\langle P, \tilde{P}\rangle \in \mathcal{L}_G(S)$.

Moreover, if such $\tilde{P}$ exists and $M_1 := \langle P, \tilde{P}\rangle^\circ C_S(Y_P)$, then $M_1/C_{M_1}(Y_{M_1}) \cong SL_3(2^n)$ or $Sp_4(2^n)'$, and $[Y_{M_1}, M_1]$ is the corresponding natural module for $M_1/C_{M_1}(Y_{M_1})$.

Small World Theorem, version $p = 2$, $Y_L \leq Q$ for all $L \in \mathcal{L}(S)$.(U.Meierfrankenfeld, B. Stellmacher)

Theorem (Baumann characteristic 2?)

One of the following holds:

- *We have non E-uniqueness*
- *For all $P \in \mathcal{P}_G(S)$ we have that $O^2(P) = q^2 L_2(q)'$, $q = 2^n$.*
- *There exists a unique $P \in \mathcal{P}_G(S)$ with $P^\circ \not\leq \tilde{C}$, and a unique $\tilde{P} \in \mathcal{P}_G(S)$ with $\tilde{P} \not\leq N_G(P^\circ)$ and $O_p(\langle P, \tilde{P} \rangle) \neq 1$. Moreover,*
  - *Let $R \in \mathcal{L}_G(P)$. Then Y_R is a natural $SL_n(q)$-module for R°, where $n \geq 2$.*
- *There exist $P_1, P_2 \in \mathcal{P}_G(S)$ such that $P_2 \leq ES$ and $O_p(\langle P_1, P_2 \rangle) = 1$.*

To do

Let $P_1, P_2 \in \mathcal{P}_G(S)$, $P_1 \not\leq \tilde{C}$, $P_2 \leq \tilde{C}$ and $O_2(\langle P_1, P_2 \rangle) = 1$.

$P_1^\circ / O_2(P_1^\circ) \cong L_2(q)$, $q = 2^n$, and Y_{P_1} the natural module.

$C_{Y_{P_1}}(S \cap P_1^\circ)$ normal in P_2.

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$b = 2$ open.