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Time-time correlation for the South polar region of the Aztec diamond

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arXiv:1602.00486 with H. Spohn
arXiv:1807.02982 with A. Occelli

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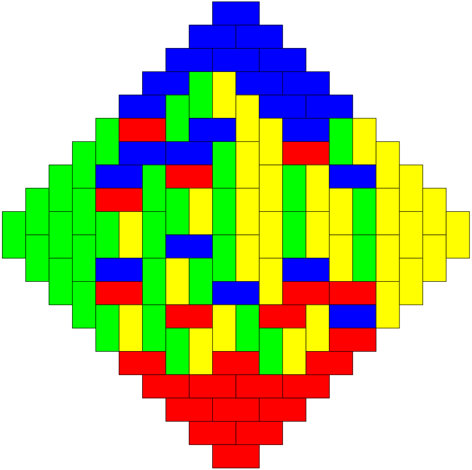
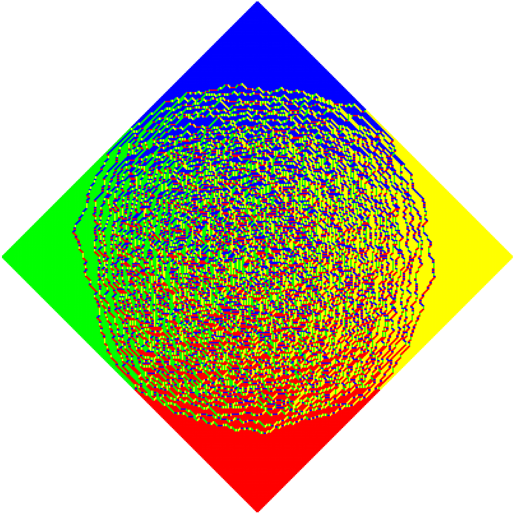
The Aztec diamond



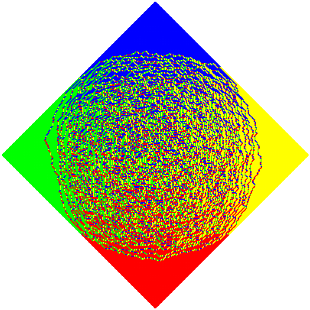
An Aztec diamond of size $N = 10$

Figure by Sunil Chhita



An Aztec diamond of size $N = 200$

Figure by Sunil Chhita



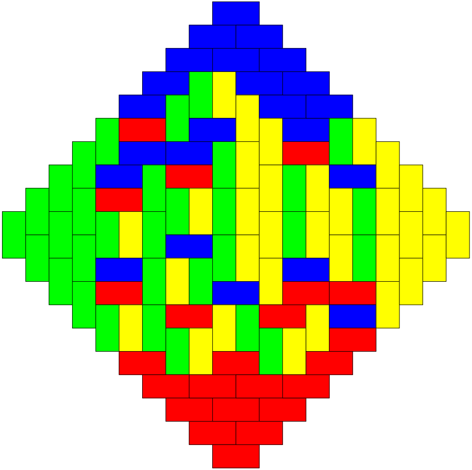
The border of the random region, as the size $N \to \infty$:

- has a circular limit shape

Jockush, Propp, Shor'98

- the border of red frozen region has fluctuations $\mathcal{O}(N^{1/3})$ and (GUE) Tracy-Widom distributed
- As a process, it converges to the Airy$_2$ process on the $(N^{2/3}, N^{1/3})$ scale

Johansson'05



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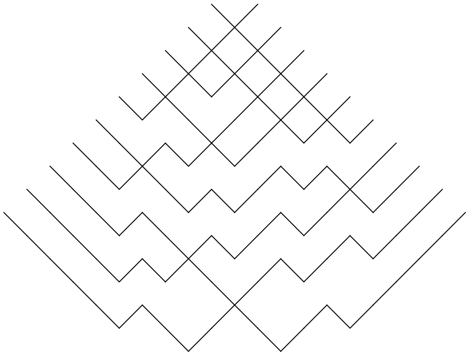
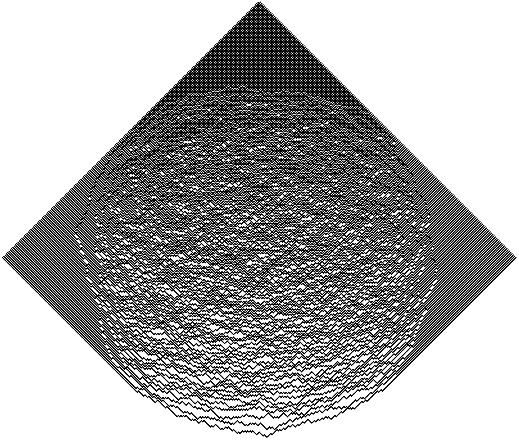


Lines of an Aztec diamond of size $N = 10$

Figure by Sunil Chhita



Lines of an Aztec diamond of size $N = 200$

Figure by Sunil Chhita

- Let $x \mapsto h(x,N)$ be the bottom curve of the Aztec diamond of size N (the origin is the south corner of the Aztec diamond).
- The rescaled height function is given by

$$h_N^{\text{resc}}(u) = \frac{h(2^{-1/6}uN^{2/3},N) - N(1-1/\sqrt{2})}{-2^{-5/6}N^{1/3}}$$

- Asymptotic results:

$$\lim_{N\to\infty} \mathbb{P}(h_N^{\text{resc}} \leq s) = F_{\text{GUE}}(s),$$

with F_{GUE} the (GUE) Tracy-Widom distribution and

$$\lim_{N\to\infty} h_N^{\text{resc}} = \mathcal{A}_2(u) - u^2,$$

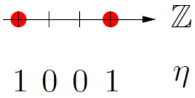
with $\mathcal{A}_2$ the Airy$_2$ process.

- For the uniform measure, an Aztec diamond of size N can be obtained from the one of size $N-1$ by the well-known shuffling algorithm Elkies,Kuperbert,Larsen,Propp'92
- This gives a discrete time Markov chain

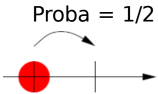
- TASEP: Totally Asymmetric Simple Exclusion Process

- **Configurations**

$$\eta = \{\eta_j\}_{j\in\mathbb{Z}},\ \eta_j = \begin{cases} 1, & \text{if } j \text{ is occupied,} \\ 0, & \text{if } j \text{ is empty.} \end{cases}$$

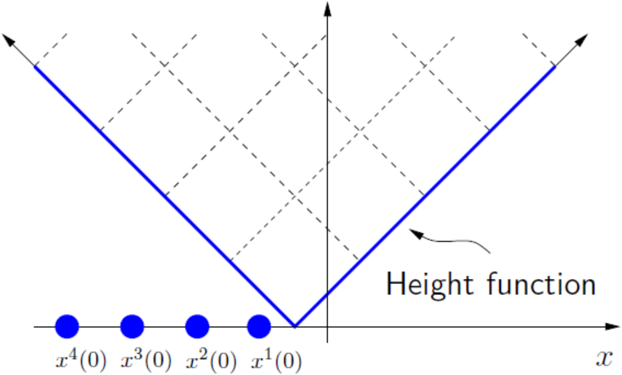


- **Dynamics**: discrete time parallel update
Select all particles whose right neighboring site is empty. Independently move them by one to the right with probability $1/2$.



$\Rightarrow$ Particles are ordered: position of particle k is $x^k(N)$

- Step initial condition is $x^k(0) = -k$, $k \geq 1$.
- Height function $h^{\text{TASEP}}(x, N)$ with $h^{\text{TASEP}}(x, 0) = |x|$.



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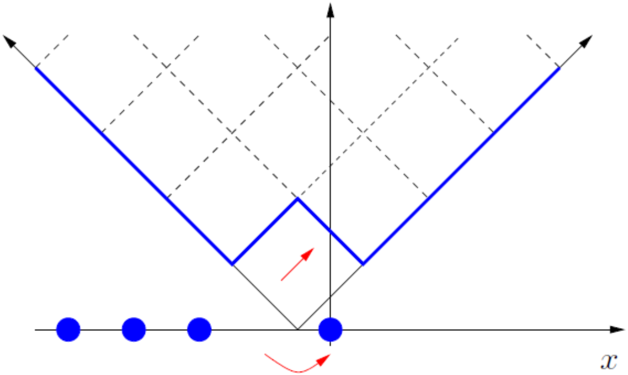

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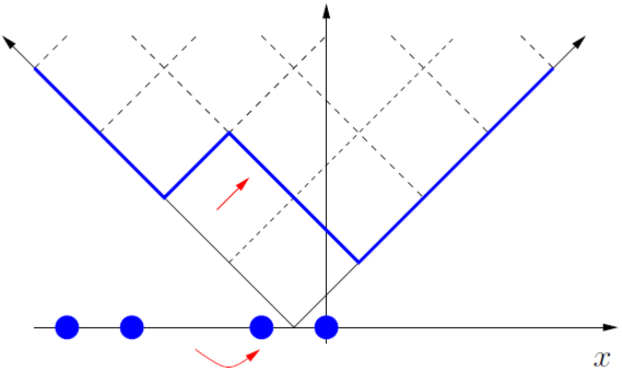

TASEP and height function

$\Rightarrow$ Particles are ordered: position of particle k is $x^k(N)$

- Step initial condition is $x^k(0) = -k$, $k \geq 1$.
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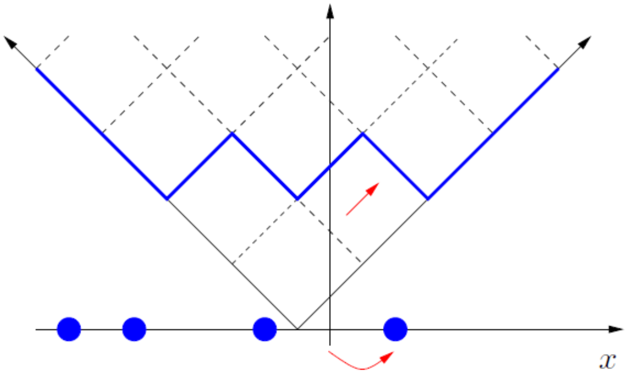


- The Aztec height function and the discrete time TASEP height function are the same object:

$$h^{\mathrm{TASEP}}(x,N) = h(x,N)$$

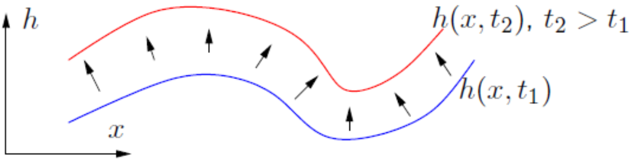
also as process in time.

- The distribution of $h(x,N)$ can be also written in terms of a last passage percolation with geometric weights.

- Spatial correlations are governed by the Airy$_2$ process.
  Q: What can one say about space-time correlations of the height function?

The Kardar-Parisi-Zhang universality class

- Surface described by a height function $h(x,t)$, $x \in \mathbb{R}$ the space, $t \in \mathbb{R}$ the time. Set wlog $h(0,0) = 0$.



- Models with local growth + smoothing mechanism

$\Rightarrow$ macroscopic growth velocity v is a function of the slope only:

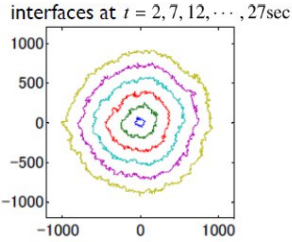
$$\partial_t h = v(\nabla h)$$

In terms of $\rho = \nabla h$ we have the PDE

$$\partial_t \rho - \nabla(v(\rho)) = 0$$

- KPZ class $\leftrightarrow v''(\nabla h) \neq 0$.
- Limit shape:

$$h_{\text{macro}}(v) := \lim_{t\to\infty} \frac{h(vt,t)}{t}$$



- Fluctuation exponent: $1/3$
- Spatial correlation exponent: $2/3$
- If $\{(x = vt, t), t \geq 0\}$ is a characteristic line of

$$\partial_t \rho - \nabla(v(\rho)) = 0$$

$\Rightarrow$ Rescaled height function around macroscopic position v,

$$h_t^{\text{resc}}(\xi) = \frac{h(vt + \xi t^{2/3}, t) - t h_{\text{macro}}(v + \xi t^{-1/3})}{t^{1/3}}$$

Limit processes at fixed time

$$h_t^{\mathrm{resc}}(\xi) = \frac{h(vt + \xi t^{2/3}, t) - t h_{\mathrm{macro}}(v + \xi t^{-1/3})}{t^{1/3}}$$

- Universality conjecture: take a v such that $\frac{d}{dv} h_{\mathrm{macro}}(v)$ exists. Then

$$\lim_{t\to\infty} h_t^{\mathrm{resc}}(\xi) = \kappa_v \mathcal{A}(\xi/\kappa_h)$$

  with κ_v, κ_h model-dependent coefficients (depending on v)
- The limit process $\mathcal{A}$ still depends on subclasses of initial conditions/boundary conditions

Analysis of exactly solvable models gives

- curved limit shape

$$\lim_{t\to\infty} h_t^{\mathrm{resc}}(\xi) = \kappa_v \mathcal{A}_2(\xi/\kappa_h)$$

with $\mathcal{A}_2$ the Airy$_2$ process.

- flat limit shape with non-random initial condition

$$\lim_{t\to\infty} h_t^{\mathrm{resc}}(\xi) = \kappa_v \mathcal{A}_1(\xi/\kappa_h)$$

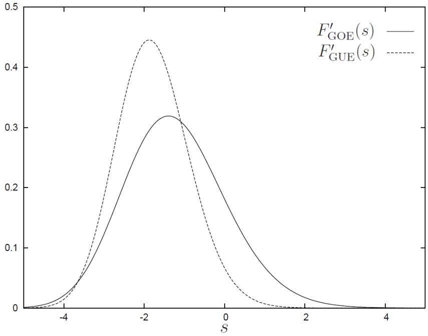
with $\mathcal{A}_1$ the Airy$_1$ process.

Borodin,Ferrari,Johansson,Prähofer,Sasamoto,Spohn'03-07

One-point distribution

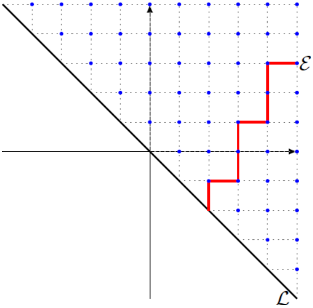
- $\mathbb{P}(\mathcal{A}_2(\xi) \leq s) = F_{\mathrm{GUE}}(s)$
- $\mathbb{P}(\mathcal{A}_1(\xi) \leq s) = F_{\mathrm{GOE}}(2s)$

are called the GUE/GOE Tracy-Widom distribution functions, discovered in random matrix theory `Tracy,Widom'94-'96`



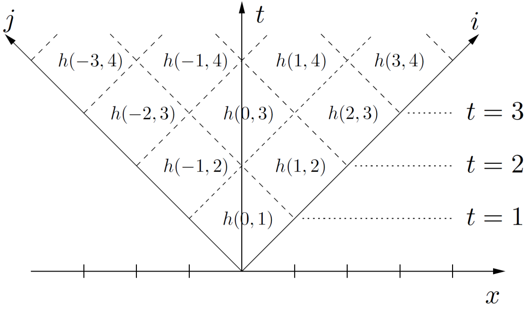
Beyond spatial correlations: space-time scaling

- Set $\mathcal{L} = \{i + j = 0\}$ (or $\mathcal{L} = \{(0,0)\}$) and end-point $\mathcal{E}$
- $\{\omega_{i,j}\}_{(i,j)>\mathcal{L}}$ iid. $\exp(1)$ random variables
- On $\mathcal{L}$ we can add some random variables (initial condition)
- Directed path π composed of $\rightarrow$ and $\uparrow$ s.t. $\pi(0) \in \mathcal{L}$ and $\pi(n) \in \mathcal{E}$
- Last passage time $L_{\mathcal{L}\to\mathcal{E}} = \max_{\substack{\pi: A\to E \\ A\in\mathcal{L}, E\in\mathcal{E}}} \sum_{0\leq k\leq n} \omega_{\pi(k)}$



- Illustration for $\mathcal{L} = \{(0,0)\}$.
  One can define the height function at time t by

$$x \mapsto h(x,t) := L_{\mathcal{L}\to(t+1+x)/2,(t+1-x)/2}$$



- This corresponds to the dynamics defined by

$$h(x,t+1) = \max\{h(x-1,t), h(x,t), h(x+1,t)\} + w(x,t)$$

with $w(x,t) = \omega((t+1+x)/2, (t+1-x)/2)$.

- Cut at $i+j=t$: height function at fixed time t
- Cut at $i=j$, i.e., $x=0$ is a characteristic direction

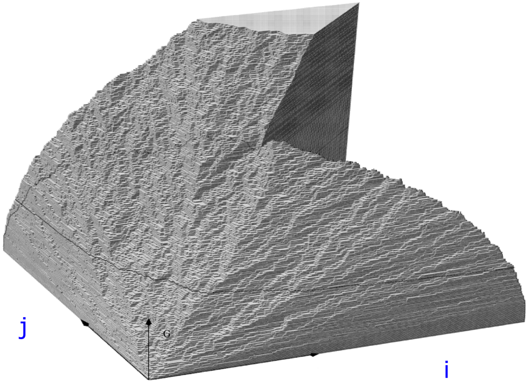
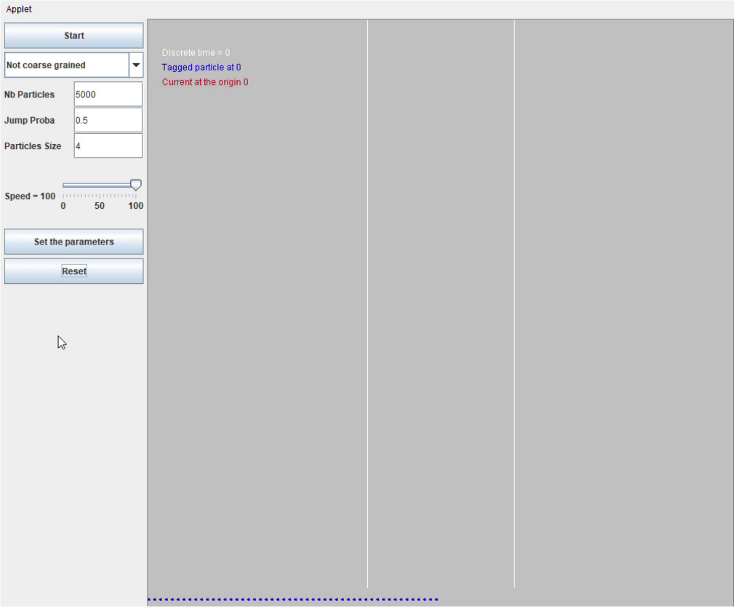
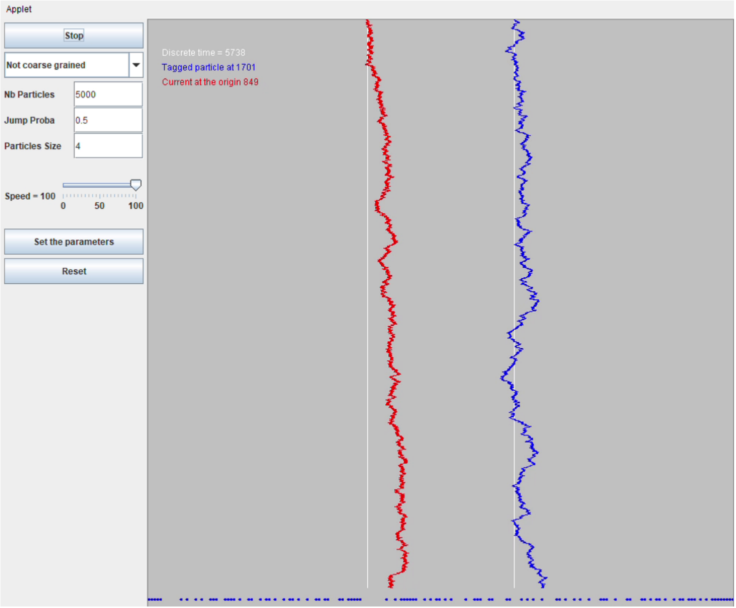


Figure by Michael Prähofer

Different space-time cuts

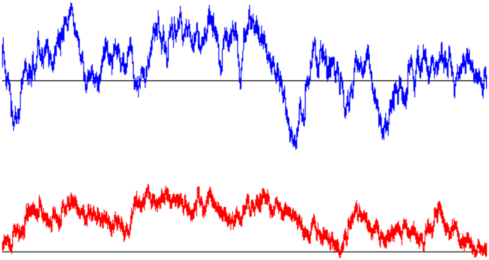



Applet
Stop
Not coarse grained
Nb Particles
5000
Jump Proba
0.5
Particles Size
4
Speed = 100
0
50
100
Set the parameters
Reset
Discrete time = 5738
Tagged particle at 1701
Current at the origin 849

Height function not along a characteristic line

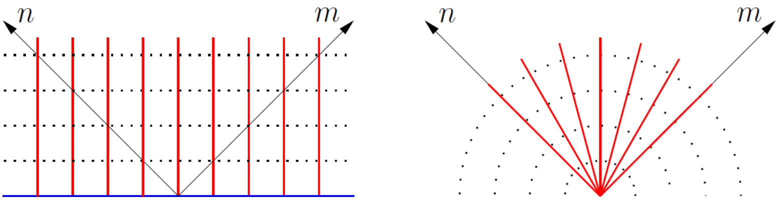
vs.

Height function along a characteristic line



- Spatial correlation length is $\mathcal{O}(t^{2/3})$
- For space-time points on a characteristic line: non-trivial correlations over macroscopic scale, i.e. $\mathcal{O}(t)$

Ferrari'08;Corwin,Ferrari,Péché'10



Characteristic lines for the point-to-line(s) (left) and for point-to-point (right)

Beyond spatial correlations: results

- The initial height function is h_0 on $\mathcal{L}$
- Point-to-point: $L^{\bullet}(m,n)$ $\quad \mathcal{L} = \{(0,0)\} \quad h_0 = 0$
- Point-to-line (flat IC): $L^{\diagdown}(m,n)$ $\quad \mathcal{L} = \{i+j=0\} \quad h_0 = 0$

- The initial height function is h_0 on $\mathcal{L}$
- Point-to-point: $L^{\bullet}(m,n)$ $\mathcal{L}=\{(0,0)\}$ $h_0=0$
- Point-to-line (flat IC): $L^{\diagdown}(m,n)$ $\mathcal{L}=\{i+j=0\}$ $h_0=0$
- Point-to-random line (random IC): $L^{\sigma}(m,n)$
  $\mathcal{L}=\{i+j=0\}$,

$$h_0(x,-x)=\sigma\begin{cases}\sum_{k=1}^{x}(X_k-Y_k) & x\geq 1\\ 0 & x=0\\ -\sum_{k=x+1}^{0}(X_k-Y_k) & x\leq -1\end{cases}$$

$\{X_k,Y_k\}_{k\in\mathbb{Z}}$ i.i.d. Exp(1/2)

- $\sigma=0$ is the flat IC, $\sigma=1$ is the stationary IC.

- For the mentioned cases $\{(0,t), t \geq 0\}$ is a characteristic line
- We are interested in the limit process

$$(u,\tau) \mapsto \mathcal{X}^\star(u,\tau) = \lim_{t\to\infty} \frac{L^\star(\tau t + u(2t)^{2/3}, \tau t - u(2t)^{1/3}) - 4\tau t}{2^{4/3} t^{1/3}}$$

- Fixed time known results: $\mathcal{A}^\star(u) = \mathcal{X}^\star(u,1)$
  - $\mathcal{A}^\bullet(u) = \mathcal{A}_2(u) - u^2$ Prähofer,Spohn'02
  - $\mathcal{A}^\diagdown(u) = \mathcal{A}_1(u)$ Sasamoto'05
  - $\mathcal{A}^{\sigma=1}(u) = \mathcal{A}_{\rm stat}(u)$ Baik,Ferrari,Péché'09
  - $\mathcal{A}^\sigma(0) = \max_{v\in\mathbb{R}}\{\sqrt{2}\sigma B(v) + \mathcal{A}_2(v) - v^2\}$ Chhita,Ferrari,Spohn'17
  - $\xi_{\rm BR} = \mathcal{A}^{\sigma=1}(0) = \max_{v\in\mathbb{R}}\{\sqrt{2}B(v) + \mathcal{A}_2(v) - v^2\}$, with $\xi_{\rm BR}$ the Baik-Rains distribution function Baik,Rains'00

- Restrict here to $u = 0$ (for the talk only). The rescaled process is

$$\tau \mapsto \mathcal{X}^\star(\tau) = \lim_{t\to\infty} \frac{L^\star(\tau t, \tau t) - 4\tau t}{2^{4/3} t^{1/3}}$$

- Goal: determine the covariance of the process at two times:

$$C^\star(\tau) = \mathrm{Cov}(\mathcal{X}^\star(\tau), \mathcal{X}^\star(1))$$

with $\star = \text{curved, flat, or random.}$

Theorem

For the stationary case, i.e., $\star = \sigma$ *with* $\sigma = 1$,

$$\mathrm{Cov}\left(\mathcal{X}^{\star}(\tau), \mathcal{X}^{\star}(1)\right) = \frac{1 + \tau^{2/3} - (1-\tau)^{2/3}}{2} \mathrm{Var}\left(\xi_{\mathrm{BR}}\right)$$

- But, the process is not a fractional Brownian motion
- Open question: what is the time-time process? Is it related with fractional Brownian motion with Hurst parameter $1/3$?

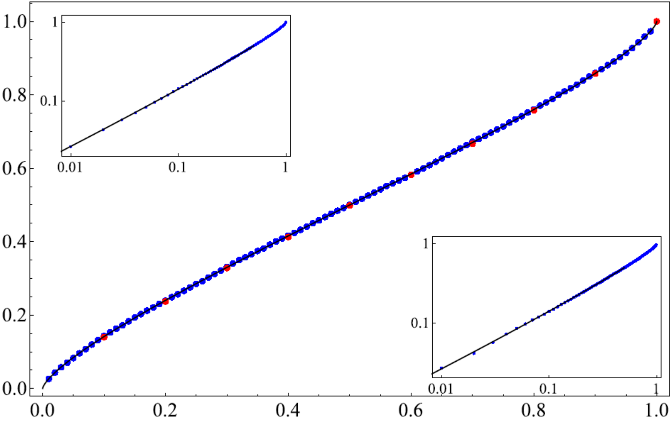


Figure: Plot of $\tau \mapsto \mathrm{Cov}(\mathcal{X}^{\mathrm{stat}}(\tau), \mathcal{X}^{\mathrm{stat}}(1)) / \mathrm{Var}(\mathcal{X}^{\mathrm{stat}}(1))$. The top-left inset is the log-log plot around $\tau = 0$ and the right-bottom inset is the log-log plot around $\tau = 1$. The fit is made with the function $\tau \mapsto \frac{1}{2}(1 + \tau^{2/3} - (1 - \tau)^{2/3})$. Ferrari,Spohn'16

Theorem (Universal behavior for $\tau \to 1$)

For $\star \in \{\bullet, \diagdown, \sigma\}$ the covariance of the limiting height function for $\tau \to 1$ is

$$\begin{aligned}\mathrm{Cov}\left(\mathcal{X}^\star(\tau), \mathcal{X}^\star(1)\right) = {} & \frac{1+\tau^{2/3}}{2}\mathrm{Var}\left(\mathcal{X}^\star(1)\right) \\ & - \frac{(1-\tau)^{2/3}}{2}\mathrm{Var}\left(\xi_{\mathrm{BR}}\right) + \mathcal{O}(1-\tau)^{1^-}.\end{aligned}$$

Analogue statement for the Aztec diamond

- Space-time scaling

$$h_N^{\mathrm{resc}}(u,\tau) = \frac{h(2^{-1/6} u N^{2/3}, \tau N) - \tau N (1 - 1/\sqrt{2})}{-2^{-5/6} N^{1/3}}$$

- Let $H(\tau) = \lim_{N\to\infty} h_N^{\mathrm{resc}}(0,\tau)$. Then the result from LPP would rewrites as

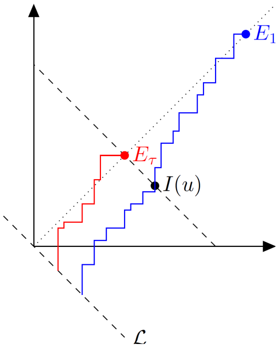
$$\begin{aligned}\mathrm{Cov}\left(H(\tau), H(1)\right) &= \frac{1+\tau^{2/3}}{2}\,\mathrm{Var}\left(\xi_{\mathrm{GUE}}\right) \\ &\quad - \frac{(1-\tau)^{2/3}}{2}\,\mathrm{Var}\left(\xi_{\mathrm{BR}}\right) + \mathcal{O}(1-\tau)^{1-\delta}\end{aligned}$$

- Consider two paths with ending points

$$E_\tau = (\tau t, \tau t) \quad \text{and} \quad E_1 = (t, t).$$

- Concatenation property: let $I(u) = \tau t(1,1) + u\,(2\tau)^{2/3}(1,-1)$, then

$$L^\star(E_1) = \max_{u\in\mathbb{R}}\{L^\star(I(u)) + L^\bullet_{I(u)\to E_1}\}$$



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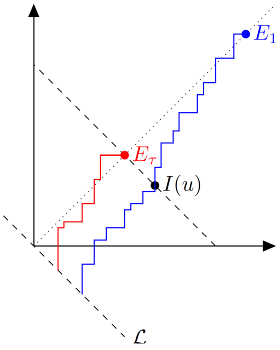

- Concatenation property: let $I(u) = \tau t(1,1) + u\,(2\tau)^{2/3}(1,-1)$, then

$$L^\star(E_1) = \max_{u\in\mathbb{R}}\{L^\star(I(u)) + L^\bullet_{I(u)\to E_1}\}$$

- Taking $t \to \infty$ we gets

$$\mathcal{X}^\star(1) = \max_{u\in\mathbb{R}}\{\tau^{1/3}\mathcal{A}^\star(\tau^{-2/3}u) + (1-\tau)^{1/3}\mathcal{A}^\bullet((1-\tau)^{-2/3}u)\}$$

and

$$\mathcal{X}^\star(\tau) = \tau^{1/3}\mathcal{A}^\star(0)$$

- Use the decomposition

$$\mathrm{Cov}(\mathcal{X}^\star(1),\mathcal{X}^\star(\tau)) = \frac{1}{2}\,\mathrm{Var}(\mathcal{X}^\star(1))+\frac{1}{2}\,\mathrm{Var}(\mathcal{X}^\star(\tau))-\frac{1}{2}\,\mathrm{Var}(\mathcal{X}^\star(1)-\mathcal{X}^\star(\tau))$$

- Thus need to control the variance of

$$\mathcal{X}^\star(1)-\mathcal{X}^\star(\tau) = \max_{u\in\mathbb{R}}\{\tau^{1/3}[\mathcal{A}^\star(\tau^{-2/3}u)-\mathcal{A}^\star(0)]+(1-\tau)^{1/3}\mathcal{A}^\bullet((1-\tau)^{-2/3}u)\}$$

- Recall: $\mathcal{A}^\bullet(u)=\mathcal{A}_2(u)-u^2$. With $u=(1-\tau)^{2/3}v$:

$$\mathcal{X}^\star(1)-\mathcal{X}^\star(\tau) = (1-\tau)^{1/3}\max_{v\in\mathbb{R}}\{(\tfrac{\tau}{1-\tau})^{1/3}[\mathcal{A}^\star((\tfrac{1-\tau}{\tau})^{2/3}v)-\mathcal{A}^\star(0)]+\mathcal{A}_2(v)-v^2\}$$

- Using the comparison with stationarity Cator,Pimentel'15

$$(\tfrac{\tau}{1-\tau})^{1/3}[\mathcal{A}^\star((\tfrac{1-\tau}{\tau})^{2/3}v)-\mathcal{A}^\star(0)]\simeq\sqrt{2}B(v)$$

as $\tau\to 1$, i.e.,

$$\mathcal{X}^\star(1)-\mathcal{X}^\star(\tau)\simeq(1-\tau)^{1/3}\max_{v\in\mathbb{R}}\{\sqrt{2}B(v)+\mathcal{A}_2(v)-v^2\}\overset{(d)}{=}\xi_{\rm BR}$$

- Using (exponential) tail estimates on the $\mathcal{X}^\star(1)$ (all of them can be obtained from the point-to-point tails with some work) we prove

$$\mathrm{Var}(\mathcal{X}^\star(1)-\mathcal{X}^\star(\tau))=(1-\tau)^{2/3}\,\mathrm{Var}(\xi_{\rm BR})+\mathcal{O}((1-\tau)^{1^-})$$

Point-to-point: numerics

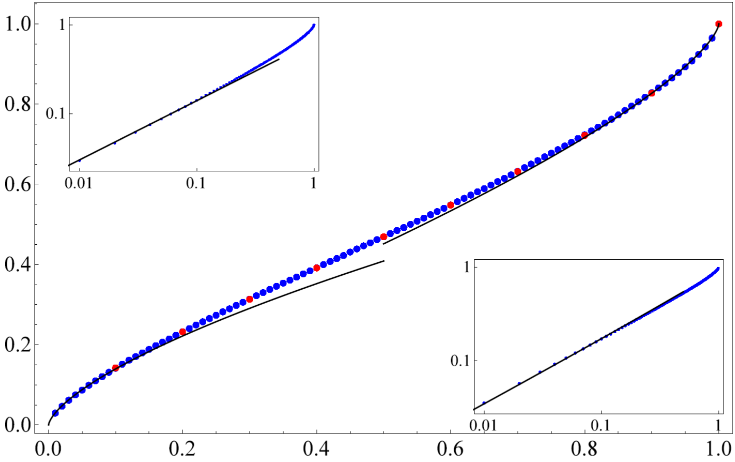

Figure: Plot of $\tau \mapsto \mathrm{Cov}(\mathcal{X}^\bullet(\tau), \mathcal{X}^\bullet(1))/\operatorname{Var}(\mathcal{X}^\bullet(1))$.
$\mathrm{Cov}(\mathcal{X}^\bullet(\tau), \mathcal{X}^\bullet(1)) \sim \tau^{2/3}$ for $\tau \to 0$.

Ferrari,Spohn'16; Ferrari,Occelli'18; Basu,Ganguly'18

Prefactor in front of $\tau^{2/3}$ is known LeDoussal'17,Johansson'19

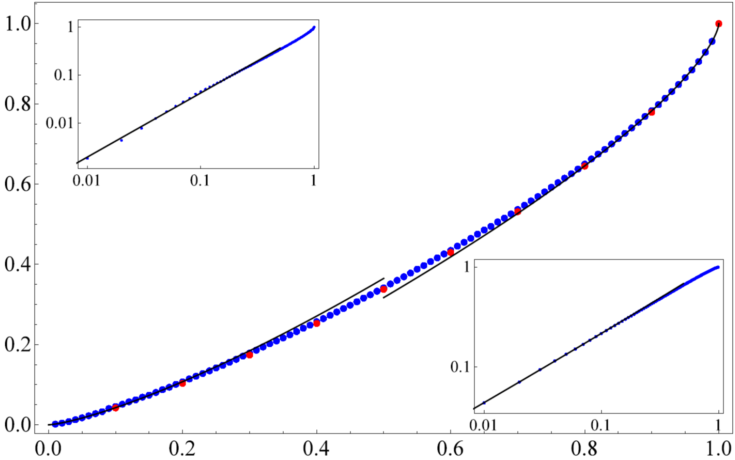

Figure: Plot of $\tau \mapsto \mathrm{Cov}(\mathcal{X}^{\diagdown}(\tau), \mathcal{X}^{\diagdown}(1)) / \mathrm{Var}(\mathcal{X}^{\diagdown}(1))$.
$\mathrm{Cov}(\mathcal{X}^{\diagdown}(\tau), \mathcal{X}^{\diagdown}(1)) \sim \tau^{4/3}$ for $\tau \to 0$. Ferrari,Spohn'16

Experimental results:

- Takeuchi (2012-2016): Experiments on turbulent liquid crystals and off-lattice Eden simulations

Mathematical results

- Baik-Liu (2017-), Johansson (2017-2018): Joint distribution function at two times (point-to-point)
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- Basu-Ganguly (2018): $O(\tau^{2/3})$ for $\tau \to 0$ and $O((1-\tau)^{2/3})$ for $\tau \to 1$ for point-to-point. Uses less inputs from exactly solvable (no Airy processes); estimates uniform in t.
- Corwin-Ghosal-Hammond (2019): similar results as Basu-Ganguly for KPZ equation with narrow-wedge initial condition