Some canonical metrics on four manifolds: rigidity and existence

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Canonical Riemannian metrics

"It is geometers dream to find a canonical metric g_{best} on a given smooth manifold M so that all topology of M will be captured by geometry." [M. Gromov]

(M^n, g) smooth Riemannian manifold, $\dim(M^n) = n \geq 2$, $\partial M = \emptyset$.

g metric $\rightsquigarrow$ Riemann ($Riem_g$), Ricci (Ric_g) and scalar curvature (R_g)

In coordinates:

$$Riem_g = (Riem)_{ijkl} \overset{\text{trace}}{\longrightarrow} Ric_g = (Ric)_{ik} = g^{jl}(Riem)_{ijkl} \overset{\text{trace}}{\longrightarrow} R_g = g^{ik}(Ric)_{ik}$$

- Constant curvature: $Riem_g = \lambda g \owedge g$ Space forms
- Constant Ricci curvature: $Ric_g = \lambda g$ Einstein metrics
- Constant scalar curvature: $R_g = \lambda$ Yamabe metrics

* G. Catino, P. Mastrolia, *A potential generalization of some canonical Riemannian metrics* , Ann. Glob. Anal. Geom., to appear.

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Einstein metrics I: variational point of view

Canonical metrics as **critical points** of curvature functionals. Let M closed,

$$\mathfrak{S}(g) = \mathrm{Vol}_g(M)^{-\frac{n-2}{n}} \int_M R_g \, dV_g \qquad \text{Einstein-Hilbert functional}$$

g is **critical** for $\mathfrak{S}(g) \iff g$ is Einstein, i.e. $Ric_g = \lambda \, g$, $\lambda \in \mathbb{R}$.

- If $n = 3$, then Einstein metrics have constant curvature.
- If $n = 4$, it is well known that there are topological obstructions to the existence of an Einstein metric (e.g. Hitchin-Thorpe: $\chi(M) \geq \frac{3}{2}|\tau(M)|$).
- If $n > 4$, still unknown.

On the other hand, the constrained problem in a conformal class (Yamabe problem) is unobstructed. More precisely the Yamabe invariant

$$\mathcal{Y}(M, [g]) = \inf_{\tilde{g} \in [g]} \mathfrak{S}(\tilde{g}) = \tfrac{4(n-1)}{n-2} \inf_{u \in W^{1,2}(M)} \frac{\int_M |\nabla u|^2 \, dV_g + \frac{n-2}{4(n-1)} \int_M R \, u^2 \, dV_g}{\left(\int_M |u|^{2n/(n-2)} \, dV_g\right)^{(n-2)/n}}$$

is always attained in *every* conformal class $[g]$.

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Einstein metrics II: the Weyl tensor

Let (M^n, g) be an Einstein manifold, i.e. $Ric_g = \lambda g$, for some $\lambda \in \mathbb{R}$ [Besse]. In particular, by tracing, the scalar curvature is constant $R_g = n\lambda$. By the decomposition of the curvature tensor

$$Riem_g = \frac{R_g}{2n(n-1)} (g \owedge g) + Weyl_g$$

So, all the *geometric* information of an Einstein manifold are contained in the **Weyl tensor**. In particular, if $Weyl_g = 0$, then (M^n, g) is a space form.

Some well known facts:

- In dimension $n = 3$ $Weyl_g = 0$, so every Einstein manifold is a space form.
- The Weyl tensor is **totally trace free**, i.e. $g^{ik} W_{ijkl} = 0$.
- Tracing the second Bianchi identity for $Riem_g$: $\nabla_t R_{ijkl} + \nabla_l R_{ijtk} + \nabla_k R_{ijlt} = 0$ and using the decomposition, we get that the Weyl tensor has **zero divergence**, i.e. $\nabla_t W_{ijkt} = 0$ (harmonic Weyl curvature).
- With some work, one can show that the Weyl tensor satisfies the **second Bianchi Identity**

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Einstein metrics III: Bochner-Weitzenböch formula

Proposition (Derdzinski, '83)

Let (M^4, g) be a four dimensional Einstein manifold. Then

$$\Delta W_{ijkl} = \tfrac{1}{2} R_g W_{ijkl} - 4 W_{ipkq} W_{pjql} - W_{klpq} W_{pqij} \, .$$

In particular,

$$\tfrac{1}{2}\Delta |Weyl_g|^2 = |\nabla Weyl_g|^2 + \tfrac{1}{2} R_g |Weyl_g|^2 - 3 W_{ijkl} W_{ijpq} W_{klpq} \, .$$

More in general, the last formula holds on four-manifolds with harmonic Weyl curvature.

Proof: Taking the divergence of the second Bianchi identity for $Weyl_g$ and commuting, we get

$$\begin{aligned} 0 &= \nabla_t \nabla_t W_{ijkl} + \nabla_t \nabla_l W_{ijtk} + \nabla_t \nabla_k W_{ijlt} \\ &= \Delta W_{ijkl} + \nabla_l \nabla_t W_{ijtk} + \nabla_k \nabla_t W_{ijlt} + Riem_g * Weyl_g \\ &= \Delta W_{ijkl} + Riem_g * Weyl_g \end{aligned}$$

Since $Riem_g = (R_g/24)(g \owedge g) + Weyl_g$ and $Weyl_g$ is trace free $\overset{\cdots}{\Longrightarrow}$ $\square$

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More in general, the last formula holds on four-manifolds with harmonic Weyl curvature.

Proof: Taking the divergence of the second Bianchi identity for $Weyl_g$ and commuting, we get

$$\begin{aligned} 0 &= \nabla_t \nabla_t W_{ijkl} + \nabla_t \nabla_l W_{ijtk} + \nabla_t \nabla_k W_{ijlt} \\ &= \Delta W_{ijkl} + \nabla_l \nabla_t W_{ijtk} + \nabla_k \nabla_t W_{ijlt} + Riem_g * Weyl_g \\ &= \Delta W_{ijkl} + Riem_g * Weyl_g \end{aligned}$$

Since $Riem_g = (R_g/24)(g \owedge g) + Weyl_g$ and $Weyl_g$ is trace free $\overset{\dots}{\Longrightarrow}$ □

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Now let (M^4, g) be a closed Einstein manifolds with positive (constant) scalar curvature $R_g > 0$. We have

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By maximum principle, if $\|Weyl_g\|_\infty \leq c\,R_g$, for some sufficiently small c, then $Weyl = 0$. Thus, (M^4, g) is isometric to a quotient of the round sphere $\mathbb{S}^4$. In fact, one can get the same conclusion with an **integral pinching** assumption. Namely, we have

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Every four dimensional closed Einstein manifold (M^4, g) satisfying

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Bochner type formula of high order

Generally speaking, we would like to understand more properties on the **Weyl tensor** of Einstein manifolds (in dimension four). We start from the following simple observation: if a smooth function u satisfies a semilinear equation $\Delta u = f(u)$ on a four dimensional Einstein manifold, then the classical Bochner formula becomes

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Thus, under suitable assumptions, one can deduce Liouville type results for this class of PDEs. To this aim we computed a **Bochner-Weitzenböch formula** for the covariant derivatives of the Weyl tensor, obtaining:

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By standard commutation rules, it is quite easy to derive "rough" Bochner type identity for the covariant derivative of Weyl, and, with some work, even a formula for the k-th covariant derivative $\nabla^k W$.

The proof of the theorem, instead, relies heavily on the *algebraic structure* of curvature operators in dimension four. In fact, on an oriented Riemannian manifold of dimension four (M^4, g), Λ^2 decomposes as the sum of two subbundles $\Lambda^2 = \Lambda^+ \otimes \Lambda^-$, which are the eigenspaces of the Hodge operator $\star : \Lambda^2 \to \Lambda^2$ corresponding respectively to the eigenvalue ± 1. Since the Weyl tensor acts on Λ^2, we have the decomposition

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As a consequence, we can show the following:

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$$\int |\nabla^2 W_g^{\pm}|^2 + \tfrac{23}{12} R_g \int |\nabla W_g^{\pm}|^2 = \tfrac{5}{12} \int |W_g^{\pm}|^2 \Big(6 |W_g^{\pm}|^2 - R_g^2 \Big) \,.$$

Corollary 2 (C.-Mastrolia)

Let (M^4, g) be a four dimensional Einstein manifold with positive scalar curvature. If

$$\int |\nabla^2 W_g|^2 \leq \tfrac{1}{12} R_g \int |\nabla W_g|^2 \,,$$

then (M^4, g) is isometric to either $\mathbb{S}^4$, $\mathbb{CP}^2$ or quotients of $\mathbb{S}^2 \times \mathbb{S}^2$.

Harmonic Weyl curvature

$$\begin{array}{ccccc} \mathcal{SF} & \subset & \mathcal{E} & \subset & \mathcal{Y} \\ & & \cap & & \\ & & \mathcal{HW} & & \end{array}$$

Let M^4 be a closed manifold. A Riemannian metric g on M^4 has harmonic Weyl curvature if

$$\delta W = \nabla_t W_{ijkt} = 0$$

- Einstein metrics have harmonic Weyl curvature.
- There are topological obstructions to the existence of harmonic Weyl metrics (e.g. Bourguignon: either Einstein or $\tau(M) = 0$).
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$$\tfrac{1}{2}\Delta|W_g|^2 = |\nabla W_g|^2 + \tfrac{1}{2}R_g|W_g|^2 - 3W_{ijkl}W_{ijpq}W_{klpq}\,.$$

Actually this formula **characterizes** harmonic Weyl metrics on closed four manifolds. This follows from the integral identity [Chang-Gursky-Yang]

$$\int_M \left(|\nabla W_g|^2 - 4|\delta W_g|^2 + \tfrac{1}{2}R_g|W_g|^2 - 3W_{ijkl}W_{ijpq}W_{klpq}\right) dV_g = 0$$

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A new variational problem

From a variational point of view it seems natural to consider the quadratic scaling-invariant Riemannian functional

$$\mathfrak{D}(g) := \mathrm{Vol}_g(M)^{\frac{1}{2}} \int_M |\delta_g W_g|_g^2 \, dV_g$$

Obviously harmonic Weyl metrics are critical points (absolute minima) of $\mathfrak{D}(g)$. In the same spirit of the Yamabe problem, we define the conformal invariant

$$\mathcal{D}(M, [g]) := \inf_{\tilde{g} \in [g]} \mathfrak{D}(\tilde{g})$$

Questions:

1. What are the geometric properties of critical metrics in the conformal class for the functional $g \mapsto \mathfrak{D}(g)$?
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We have the following characterization of critical metrics in the conformal class for the functional

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A metric is critical in the conformal class for the functional $g \mapsto \mathfrak{D}(g)$ if and only if it satisfies the Weitzenböck formula

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Existence of WHW metrics

Theorem 2 (C.-Mastrolia-Monticelli-Punzo)

On every closed four-dimensional manifold there exists a weak harmonic Weyl metric.

- Aubin proved that every closed Riemannian manifold admits a constant negative scalar curvature metric. Besides this one, Theorem 2 is the only existence result of a *canonical* metric, which generalizes the Einstein condition, on *every* four-dimensional Riemannian manifold, without any topological obstructions.
- The metric in Theorem 2 is constructed as follows: first, thanks to a result of Aubin, on every four-dimensional manifold M^4 we can choose a reference metric g_0 with $|W_{g_0}|_{g_0} > 0$. Then, we prove that on (M^4, g_0) the infimum $\mathcal{D}(M, [g_0])$ is attained by a conformal metric $g \in [g_0]$, which is a weak harmonic Weyl metric. Moreover, we show that every critical point in the conformal class $[g_0]$ is necessarily a minimum point.

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In order to prove this theorem, we endow a closed four-manifolds M^4 with the metric g_0 constructed by Aubin and we consider the functional

$$\begin{aligned}\mathfrak{D}(v) &:= \mathfrak{D}(v^{-2}g_0)\\ &= \left(\int_M v^{-4}dV\right)^{\frac{1}{2}} \int_M \left(\frac{1}{4}|W|^2|\nabla v|^2 + |\delta W|^2 v^2 - (v)^2_s\, W_{sijk}W_{pijk,p}\right) dV\,,\end{aligned}$$

where all the geometric quantities are referred to g_0 and the function v belongs to the convex cone

$$H(M) := \left\{ u \in H^1(M) : u > 0 \text{ a.e. and } \int_M u^{-4}\, dV < \infty \right\}.$$

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$$H(M) := \left\{ u \in H^1(M) : u > 0 \text{ a.e. and } \int_M u^{-4}\, dV < \infty \right\}.$$

We define

$$\mathcal{D} := \inf_{u\in H(M)} \mathfrak{D}(u)\,.$$

We note that the condition $|W| > 0$ is crucial, as it implies the uniform ellipticity of the problem.

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One has

$$\mathfrak{D}(v) = \left(\int_M v^{-4} dV\right)^{\frac{1}{2}} \int_M \left(a|\nabla v|^2 + c\, v^2\right)\, dV = \left(\int_M v^{-4} dV\right)^{\frac{1}{2}} \int_M v\, Lv\, dV$$

with $a \in C^\infty(M)$, $a > 0$, $c \in C^\infty(M)$ and the uniformly elliptic self-adjoint operator L is given by

$$Lv := -\operatorname{div}(a \nabla v) + c\, v.$$

Since, by definition, $\mathfrak{D}(v) \geq 0$, we get

$$\lambda_1 := \inf_{u \in H^1(M),\, u \not\equiv 0} \frac{\int_M \left(a|\nabla u|^2 + c\, u^2\right)\, dV}{\int_M u^2\, dV} \geq 0.$$

By standard elliptic theory, there exists a smooth, positive, first eigenfunction φ_1 of L solution of $L\varphi_1 = \lambda_1 \varphi_1$.

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We have the following (strong) maximum principle.

Lemma 1

Let $\lambda_1 > 0$. If $u \in H^1(M)$ satisfies $Lu \geq 0$ in the weak sense, then either $u = 0$ a.e. on M or $\operatorname{essinf}_M u > 0$.

Moreover, by Jensen, we can show a two-sided estimate on $\mathcal{D} = \inf_{u \in H(M)} \mathfrak{D}(u)$ in terms of λ_1

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We have

$$\mathrm{Vol}(M)^{\frac{3}{2}}\, \lambda_1 \leq \mathcal{D} \leq \frac{\int_M \varphi_1^2 \, dV}{\left(\int_M \varphi_1^{-4} \, dV\right)^{\frac{1}{2}}}\, \lambda_1 \,.$$

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A variational argument shows that $u \mapsto \mathfrak{D}(u)$ admits a minimum point v in $H(M)$. Consequently, v is a (weak) solution of the Euler-Lagrange equation

$$-\frac{1}{4}\,\mathrm{div}(|W|^2\nabla v) + \left(|\delta W|^2 + \mathrm{div}(W_{sijk}W_{pijk,p})\right) v = \mathfrak{D}(v)\left(\int_M v^{-4}dV\right)^{-3/2}\frac{1}{v^5}\,,$$

which is a uniformly elliptic semilinear equation with singular nonlinearity. Here, again, all the geometric quantities are referred to g_0. Moreover, v is unique up to scaling. Hence, by standard elliptic regularity theory, $v \in C^\infty(M)$ and

$$\mathcal{D}(M,[g_0]) = \min_{0<u\in C^\infty(M)}\mathfrak{D}(u) = \min_{u\in H(M)}\mathfrak{D}(u) = \mathcal{D}\,.$$

Therefore

$$g := v^{-2}g_0$$

is a weak harmonic Weyl metric on M^4. □

Conclusion: if we choose a reference metric g_0 with $|W_{g_0}|_{g_0} > 0$, then we can find a conformal metric $g = v^{-2}g_0$, $v \in C^\infty(M)$, minimizing the functional $\mathfrak{D}(g)$.

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What happens in the degenerate case, i.e. if $|W_{g_0}|_{g_0} = 0$ somewhere in M? We can show that uniqueness (up to scaling) of smooth (C^2) solutions to the equation still holds, unless g_0 is locally conformally flat, i.e. $W_{g_0} \equiv 0$. Moreover we have the following non-existence results:

- If $|W_{g_0}|_{g_0} \equiv 0$ on some open set $\Omega \subset M$, then we can show that a smooth metric $g = v^{-2} g_0$ is critical if and only if it is locally conformally flat, i.e. $W_g \equiv 0$ on M (and thus $W_{g_0} \equiv 0$).
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- If $|W_{g_0}|_{g_0} \equiv 0$ on some open set $\Omega \subset M$, then we can show that a smooth metric $g = v^{-2} g_0$ is critical if and only if it is locally conformally flat, i.e. $W_g \equiv 0$ on M (and thus $W_{g_0} \equiv 0$).
- In addition, if $|W_{g_0}|_{g_0}(p) = 0$ at some point $p \in M$ and

$$\frac{|W_{g_0}|_{g_0}(x)}{\mathrm{dist}_{g_0}(x,p)} \to 0 \quad \text{as } \mathrm{dist}_{g_0}(x,p) \to 0,$$

then, a smooth metric $g = v^{-2} g_0$ is critical if and only if it has harmonic Weyl, i.e. $\delta_g W_g \equiv 0$ on M. As we have seen, there are topological obstructions to the existence of such a metric.

The remaining case, i.e. $|W_{g_0}|_{g_0}(p) = 0$ at some point $p \in M$ and

$$\limsup_{\mathrm{dist}_{g_0}(x,p) \to 0} \frac{|W_{g_0}|_{g_0}(x)}{\mathrm{dist}_{g_0}(x,p)} > 0$$

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Thank you.

Some references:

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