Non simple blow-up phenomena for the singular Liouville equation

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Nonlinear Geometric PDE's

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Problem (1) has been widely studied: there are many papers investigating the existence of solutions with multiple concentration as $\lambda \to 0^+$.

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H is a smooth function in $\Omega \times \Omega$.
$H(x,x)$ is the Robin's function and satisfies

$$H(x,x) \to -\infty \quad \text{as } \operatorname{dist}(x, \partial\Omega) \to 0.$$

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for some $m \geq 1$. Moreover there are distinct points $\xi_1, \ldots, \xi_m \in \Omega \setminus \{0\}$ such that (up to a subsequence)

$$\lambda V(x)e^{u_\lambda} \to 8\pi \sum_{j=1}^{m} \delta_{\xi_j} \qquad (2)$$

in the measure sense.

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in the measure sense. Besides $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_m)$ corresponds to a critical point of

$$\Psi(\boldsymbol{\xi}) = \frac{1}{2}\sum_{j=1}^{m}\left(H(\xi_j,\xi_j) + \frac{\log V(\xi_j)}{4\pi}\right) + \frac{1}{2}\sum_{\substack{j,k=1\\ j\neq k}}^{m} G(\xi_j,\xi_k) - \frac{N}{2}\sum_{j=1}^{m} G(\xi_j,0).$$

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$$\lambda \int_\Omega V(x) e^{u_\lambda} dx \to 8\pi m + 8\pi(1+N) \ \text{ as } \ \lambda \to 0$$

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- Del Pino-Esposito-Musso ('10): if $N \in \mathbb{N}$ then there exists a suitable $p \in \Omega$ (depending on λ) such that a solution blowing up at $N+1$ points at the vertices of a small polygon centered at p does exist for the problem

$$\begin{cases} -\Delta u = \lambda e^u - 4\pi N \delta_p & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}.$$

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For any $N \in \mathbb{N}$, we can associate to (4) a limiting problem of Liouville type:

$$-\Delta w = |x|^{2N} e^{w} \quad \text{in } \mathbb{R}^2, \qquad \int_{\mathbb{R}^2} |x|^{2N} e^{w(x)} dx < +\infty.$$

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All solutions of this problem are given, in complex notation, by the three-parameter family of functions

$$w_{\delta,b}(x) := \log \frac{8(N+1)^2 \delta^{2(N+1)}}{(\delta^{2(N+1)} + |x^{N+1} - b|^2)^2} \quad \delta > 0,\ b \in \mathbb{C}.$$

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The following quantization property holds:

$$\int_{\mathbb{R}^2} |x|^{2N} e^{w_{\delta,b}(x)} dx = 8\pi(N+1).$$

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has a nondegenerate maximum at 0, where $\beta_i^{N+1} = b$, $\beta_i \neq \beta_h$ for $i \neq h$.

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where $\mu \sim \frac{\sqrt{\lambda}}{|b|^{\frac{N}{N+1}}}, \quad \sqrt{\lambda |\log\lambda|} \le |b| \le \lambda^{\frac{\eta}{4(N+1)}} \sqrt{|\log\lambda|}.$

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Problem (4) is the Euler Lagrange equation of the following functional

$$I(v) = \frac{1}{2}\int_{\Omega} |\nabla v|^2 dx - \lambda \int_{\Omega} |x|^{2N} \widetilde{V}(x) e^{v} dx.$$

SKETCH OF THE PROOF

STEP 1. The variational structure

Let us set:

$$v = u + 4\pi N\, G(x, 0);$$

problem (1) is then equivalent to

$$\begin{cases} -\Delta v = \lambda |x|^{2N} \widetilde{V}(x) e^{v} & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}, \tag{4}$$

where

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By the classical Moser-Trudinger inequality we get $I \in C^1(H_0^1(\Omega))$.

STEP 2. Construction of approximate solutions.

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We set

$$W_\lambda = w_{\delta,b}(x) := \log \frac{8(N+1)^2\delta^{2(N+1)}}{(\delta^{2(N+1)} + |x^{N+1} - b|^2)^2}$$

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Consider the projections PW_λ onto the space $H_0^1(\Omega)$ of W_λ, where $P : H^1(\mathbb{R}^2) \to H_0^1(\Omega)$ is defined as

$$\Delta Pv = \Delta v \quad \text{in } \Omega, \qquad Pv = 0 \quad \text{on } \partial\Omega.$$

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The following asymptotic expansion holds:

$$PW_\lambda = W_\lambda - \log(8(N+1)^2\delta^{2(N+1)}) + 8\pi \sum_{i=0}^{N} H(x, \beta_i) + O(\delta^{2(N+1)}).$$

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We shall look for a solution of the form

$$v_\lambda = PW_\lambda + \phi_\lambda, \qquad \phi_\lambda \text{ small.}$$

STEP 3. The reduced problem.

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$$J_\lambda(b) = 8\pi(N+1)(1+\log\lambda - \log(8(N+1)^2)) + 32\pi^2\Lambda(b) + 8\pi(N+1)|b|^{2\frac{N_\lambda - N}{N+1}} + \mathrm{h.o.t.}$$

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& + 8\pi(N+1)|b|^{2\frac{N_\lambda - N}{N+1}} + \text{h.o.t.}
\end{aligned}
$$

If $N_\lambda > N$, J_λ verifies

$$
J_\lambda(\sqrt{N_\lambda - N}) > \sup\left\{ J_\lambda(b) \,\middle|\, \frac{\sqrt{N_\lambda - N}}{|\log(N_\lambda - N)|} < |b| < \sqrt{N_\lambda - N}|\log(N_\lambda - N)| \right\}.
$$

Thank you for your attention!