Bubbling nodal solutions for a large perturbation of the Moser-Trudinger equation on planar domains

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Nonlinear Geometric PDE's

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A critical problem in dimension two

Let $\Omega \subseteq \mathbb{R}^2$ be a bounded smooth domain. We want to discuss the existence of non-trivial weak solutions of the problem

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where is f a non-linearity with critical growth i.e. $f(u) \sim \lambda u e^{u^2}$ for some $\lambda > 0$.

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Theorem (Pohozaev '65, Trudinger '67, Moser '71)

- $u \in H_0^1(\Omega) \implies e^{u^2} \in L^1(\Omega)$.
- *Uniform integrability on spheres in $H_0^1(\Omega)$:*

$$\sup_{u \in H_0^1(\Omega), \|\nabla u\|_2^2 = \Lambda} \int_\Omega e^{u^2} dx < +\infty \iff \Lambda \leq 4\pi.$$

MT critical equation and its perturbations

Carleson-Chang 86, Struwe '88 Flucher 92: The supremum in the MT-inequality is attained for any $\Lambda \leq 4\pi$.

Extremal functions for the MT inequality solve MT critical equation

$$-\Delta u = \lambda u e^{u^2} \tag{MT}$$

Equation (MT) has been studied by several authors (Adimurthi, Carleson, Chang, Del Pino, Druet, Li, Malchiodi, Martinazzi, Musso, Ruf, Thizy, Yadava ...)

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Question: What happens if $\lambda u e^{u^2}$ is replaced by a similar function?
General notions of functions with **critical growth** were introduced e.g. by Atkinson-Peletier '87, Adimurthi '90, Adimurthi-Druet '01, Druet '06...

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Model nonlinearities:

$$f(u) = \lambda u e^{u^2 \pm a|u|^p} \quad \lambda > 0, a \geq 0, p \in [0, 2).$$

THE MODEL PROBLEM

We will discuss the existence of solutions for the problem

$$\begin{cases} -\Delta u = \lambda u e^{u^2+|u|^p} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \qquad (\mathcal{P}_{\lambda,p})$$

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Plan of the talk:

1. An overview of the main existence results for $(\mathcal{P}_{\lambda,p})$
2. Existence of **bubbling sign-changing solutions** on generic Ω, when λ is small and $p \in (1,2)$, $p \sim 1$.

Grossi, M., Naimen, Pistoia, *Bubbling nodal solutions for a large perturbation of the Moser-Trudinger equation on planar domains*, preprint 2019.

Positive solutions

Adimurthi '90: $(\mathcal{P}_{\lambda,p})$ admits a positive solution for any $p \in (0,2)$ if $0 < \lambda < \lambda_1$, where $\lambda_1 = \lambda_1(\Omega)$ is the first eigenvalue of $-\Delta$ on Ω.

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Idea: Consider the energy functional

$$J(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega F(u)dx, \qquad F(t) := \lambda \int_0^t s e^{s^2+s^p} ds.$$

and let $\mathcal{N}$ be the corresponding Nehari manifold

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- J satisfies the Palais-Smale condition in the energy range $(-\infty, 2\pi)$.
- If $\lambda < \lambda_1$, then $0 < \inf_{\mathcal{N}} J < 2\pi$.

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- If $\lambda < \lambda_1$, then $0 < \inf_{\mathcal{N}} J < 2\pi$.

Remarks:

- $\lambda < \lambda_1$ is necessary for the existence of a positive solution.
- The solution blows-up as $\lambda \to 0$ (Adimurthi-Druet '01).

Existence of sign-changing solutions

Adimurthi-Yadava '90: Assume $\Omega = B(0,1)$. If $0 < \lambda < \lambda_1$ and $p > 1$, then for any $k \in \mathbb{N}$, $(\mathcal{P}_{\lambda,p})$ admits a non-trivial radial weak solution with k nodal regions.

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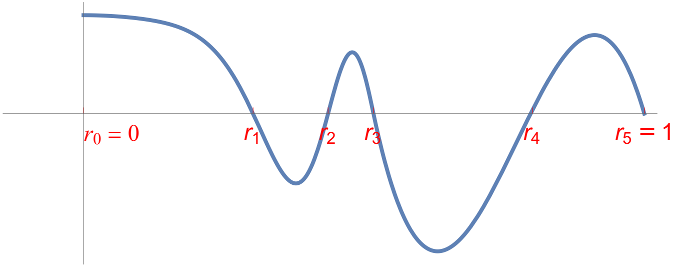
Idea: Minimize J on $\mathcal{N}_k$, where

$$\mathcal{N}_k := \{u \in H^1_{0,r} \ : \ \exists\, 0 = r_0 < r_1 < \ldots < r_k = 1 \text{ s.t. } u \in \Gamma(r_0, \ldots, r_k)\}$$

$$\Gamma(r_0, \ldots, r_k) = \{u \ : \ (-1)^i u|_{(r_i, r_{i+1})} > 0,\ u\chi_{(r_i, r_{i+1})} \in \mathcal{N}, 0 \le i \le k-1\}.$$



Sharpness of the growth condition

Adimurthi-Yadava '92: If $p \leq 1$, then there exists $\lambda_{AY} = \lambda_{AY}(p) > 0$ such that $(\mathcal{P}_{\lambda,p})$ has **no radial sign-changing solution** when $\Omega = B(0,1)$ and $0 < \lambda < \lambda_{AY}$.

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Remarks:

- The case $p = 1$ defines a borderline case between existence and non-existence of radial nodal solutions on $B(0,1)$ for small λ.
- If $\Omega = B(0,1)$, one can find non-radial solutions for any $\lambda > 0$, $p \in (0,2)$. But when Ω is non-symmetric, it is not known whether $(\mathcal{P}_{\lambda,p})$ has sign-changing solutions when $0 < \lambda < \lambda_1$ and $0 \leq p \leq 1$.

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Question: For $0 < \lambda < \min\{\lambda_{AY}(1), \lambda_1\}$, what is the behavior of solutions as $p \to 1^+$?

Asymptotic of radial solutions

Assume $\Omega = B(0,1)$ and consider $(\mathcal{P}_{\lambda,p})$ with $p = 1 + \varepsilon$ and a small fixed λ. For $k \in \mathbb{N}$, $\varepsilon > 0$, let $u_\varepsilon \in \mathcal{N}_{k,\varepsilon}$ be s.t. $J_\varepsilon(u_\varepsilon) = \min_{\mathcal{N}_{k,\varepsilon}} J_\varepsilon$.

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Grossi-Naimen '18: As $\varepsilon \to 0$ we have:

- Let $0 < r_{1,\varepsilon} < \ldots < r_{k-1,\varepsilon} < r_{k,\varepsilon} = 1$ be the zeroes of u_ε, then $r_{i,\varepsilon} \to 0$ for $1 \le i \le k-1$.
- $u_\varepsilon \to (-1)^{k-1} u_0$ in $C^2_{loc}(B(0,1) \setminus \{0\})$, where u_0 is the unique positive radial solution to $(\mathcal{P}_{\lambda,1})$.
- For $i = 1, \ldots, k$ let $A_{i,\varepsilon}$ be the i-th nodal region of u_ε and let $M_{i,\varepsilon}$ be a maximum point of $|u_\varepsilon|$ on $A_{i,\varepsilon}$. Then, $\exists\ \delta_{i,\varepsilon} > 0$, s.t.

$$2u_\varepsilon(M_{i,\varepsilon})\left(u_\varepsilon(M_{i,\varepsilon} + \delta_{i,\varepsilon} r) - u_\varepsilon(M_{i,\varepsilon})\right) \to -2\log\left(1 + \frac{r^2}{8}\right)$$

in $C^1_{loc}([0,\infty))$.

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Question: If Ω is not a ball, are there solutions with a similar behavior as $p \to 1^+$? YES, $k = 2$.

Our main result

Let $\Omega \subseteq \mathbb{R}^2$ be an **arbitrary domain** and fix $0 < \lambda < \lambda_1$. Consider

$$\begin{cases} -\Delta u = \lambda u e^{u^2 + |u|^{1+\varepsilon}} =: f_\varepsilon(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \qquad (\mathcal{P}_\varepsilon)$$

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Theorem

Let u_0 be a positive solution to $(\mathcal{P}_0)$. Assume that

- (A1) *u_0 is non-degenerate i.e. $-\Delta\varphi = f_0'(u_0)\varphi$ has no non-zero weak solution $\varphi \in H_0^1(\Omega)$.*
- (A2) *u_0 has a stable critical point ξ_0 such that $u_0(\xi) > \frac{1}{2}$.*

Then we can find $\varepsilon_0 \in (0,1)$, s.t. for any $\varepsilon \in (0,1)$ problem $(\mathcal{P}_\varepsilon)$ has a solution u_ε with 2 nodal regions. Moreover, as $\varepsilon \to 0$, u_ε satisfies

- *u_ε blows-up at ξ_0 i.e. $\sup_{B_r(\xi_0)} u_\varepsilon \to +\infty \quad \forall\, 0 < r < d(\xi_0, \partial\Omega)$.*
- *$u_\varepsilon \to -u_0$ in $C^1_{loc}(\bar{\Omega} \setminus \{\xi_0\})$.*

Idea of the proof: notation and preliminaries

Strategy: Lyapunov-Schmidt reduction methods.

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- Solutions of the Liouville equation in $\mathbb{R}^2$:

$$\begin{cases} -\Delta U = e^U \\ e^U \in L^1(\mathbb{R}^2) \end{cases} \iff U = \log\left(\frac{8\delta^2}{(\delta^2 + |x-\xi|^2)^2}\right), \delta > 0, \xi \in \mathbb{R}^2.$$

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- Notation for Liouville bubbles:

$$U_{\delta,\mu,\xi} := \log\left(\frac{8\delta^2\mu^2}{(\delta^2\mu^2 + |x-\xi|^2)^2}\right)$$

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- Projected bubbles: let $(-\Delta)^{-1} : L^q \to H_0^1$ be the inverse of $-\Delta$, then

$$PU_{\delta,\mu,\xi} = (-\Delta)^{-1} e^{U_{\delta,\mu,\xi}}, \quad \delta > 0, \mu > 0, \xi \in \Omega.$$

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Remark: Del Pino, Musso, Ruf constructed bubbling positive solutions to $-\Delta u = \lambda u e^{u^2}$, when λ is small. In the 1-bubble case they take $\alpha PU_{\delta,\mu,\xi}$ as approximate solution for suitable α, δ, μ, ξ.

The approximate solution

First idea: Use $\omega = \omega_{\alpha,\delta,\mu,\xi} := \alpha P U_{\delta,\mu,\xi} - u_0$ as an approximate solution.

Question: How small is the error term

$$
\begin{aligned}
R :=& \Delta\omega + f_\varepsilon(\omega) \\
=& \lambda\omega e^{\omega^2+|\omega|^{1+\varepsilon}} - \alpha e^{U_{\delta,\mu,\xi}} - \alpha\Delta u_0?
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Choice of α and δ: For any $0 < \varepsilon \ll 1$, $\mu > 0$, $\xi \in \Omega$, with $u_0(\xi) > \frac{1}{2}$, there exist $(\alpha, \delta) \in \mathbb{R}^+ \times \mathbb{R}^+$ s. t.

$$R = O(\alpha^3 e^{U_{\delta,\mu,\xi}}) \quad \text{in } B(\xi, r\delta), r > 0.$$

Moreover

$$\alpha = \frac{1}{2} e^{-\frac{\log(2u_0(\xi)) + o(1)}{\varepsilon}} \quad \text{and} \quad \delta = e^{-\frac{1+o(1)}{8\alpha^2}}.$$

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Remark: Assumption $(A2)$ is crucial in the choice of α and δ.

A refined approximate solution

Problem: For any $\sigma > 0$ we have $R = O(\varepsilon)$ in $\Omega \setminus B(\xi, \sigma)$.

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A refined ansatz:
$(A1) \Longrightarrow$ for $\varepsilon \in (0, \varepsilon_0)$, there exists $v_\varepsilon \in H^1(\Omega)$ s.t.

$$\begin{cases} -\Delta v_\varepsilon = \lambda e^{v_\varepsilon^2 + |v_\varepsilon|^{1+\varepsilon}} & \text{in } \Omega \\ v_\varepsilon > 0 & \text{in } \Omega \\ v_\varepsilon = 0 & \text{on } \partial\Omega \\ v_\varepsilon \to u_0 \text{ in } C^1(\bar{\Omega}) & \text{as } \varepsilon \to 0. \end{cases}$$

We define

$$\omega := \alpha P U_{\delta,\mu,\xi} - v_\varepsilon.$$

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As before, for any μ, ξ and any small ε we can choose α and δ that

$$R = O(\alpha^3 e^{U_{\varepsilon,\mu,\xi}}) \text{ in } B(\xi, r\delta), r > 0$$
$$R = O(\alpha) \text{ in } \Omega \setminus B(\xi, \sigma), \sigma > 0.$$

The final ansatz

We take

$$\omega = \alpha P U_{\delta,\mu,\xi} - v_\varepsilon - \alpha w_{\varepsilon,\xi} - \alpha^2 z_{\varepsilon,\xi}$$

where

$$\begin{cases} \Delta w_{\varepsilon,\xi} + f'_\varepsilon(v_\varepsilon) w_{\varepsilon,\xi} = 8\pi f'_\varepsilon(v_\varepsilon) G_\xi & \text{in } \Omega \\ \Delta z_{\varepsilon,\xi} + f'_\varepsilon(v_\varepsilon) z_{\varepsilon,\xi} = \frac{1}{2} f''(-v_\varepsilon)(8\pi G_\xi - w_{\varepsilon,\xi})^2 & \text{in } \Omega \\ w_\varepsilon = z_\varepsilon = 0 & \text{on } \partial\Omega \end{cases}$$

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We can define a suitable norm $\|\cdot\|_*$ such that we have the global estimate

$$\|R\|_* = O(\alpha^3).$$

The linear theory

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Remark:

- $\mathcal{L}_1$ is invertible because of $(A1)$.
- $\mathcal{L}_0$ has a 3-d approximate kernel:

$$\begin{cases} \mathcal{L}_0\varphi = 0 \\ \varphi \in L^\infty(\mathbb{R}^2) \end{cases} \iff \varphi \in \mathrm{Span}\{Z_{0,\delta,\mu,\xi}, Z_{1,\delta,\mu,\xi}, Z_{2,\delta,\mu,\xi}\}$$

$$Z_{0,\delta,\mu,\xi} := \frac{\delta^2\mu^2 - |x-\xi|^2}{\delta^2\mu^2 + |x-\xi|^2}, \quad Z_{i,\delta,\mu,\xi} := \frac{2\delta\mu(x_i - \xi_i)}{\delta^2\mu^2 + |x-\xi|^2}, \ i = 1, 2.$$

The final argument

For any $0 < \varepsilon \ll 1$, $\mu > 0$ and ξ close to ξ_0, $\exists\ \varphi = \varphi_{\varepsilon,\mu,\xi} \in H_0^1(\Omega)$, and $\kappa_i = \kappa_i(\varepsilon, \mu)$, $i = 0, 1, 2$, s.t. $\|\varphi\|_{L^\infty(\Omega)} = O(\alpha^3)$ and $u = \omega + \varphi$ solves

$$-\Delta u = f_\varepsilon(u) + \sum_{i=0}^{3} \kappa_{i,\varepsilon} e^{U_{\delta,\mu,\xi}} Z_{i,\delta,\mu,\xi}$$

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1. For $i = 0, 1, 2$, $\kappa_{i,\varepsilon}$ depends continuously on μ and ξ.
2. We have the expansions

$$\kappa_{0,\varepsilon}(\mu, \xi) = 6\pi\alpha^3 \left(2 - \log(8\mu^{-2})\right) + o(\alpha^3)$$

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Conclusion:
Since $(\sqrt{8}e^{-1}, \xi_0)$ is a stable zero of $V_0(\mu, \xi) := (2 - \log(8\mu^{-2}), \nabla u_0)$, for any small ε we can find $\mu = \mu(\xi)$, $\xi = \xi(\varepsilon)$, s.t. $\kappa_{i,\varepsilon} = 0$, $i = 0, 1, 2$.

THANK YOU

FOR YOUR ATTENTION!