A double mean field approach for a curvature prescription problem

Work in progress with R. López-Soriano

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Prescribing curvatures on surfaces

We consider the following PDE on compact surface with boundary

$$\begin{cases} -\Delta v + 2K_g = 2Ke^v & \text{in } \Sigma \\ \partial_\nu v + 2h_g = 2he^{\frac{v}{2}} & \text{on } \partial\Sigma \end{cases} ; \qquad (P_{K,h})$$

- K_g is the **Gaussian curvature** associated to g;
- h_g is the **geodesic curvature** associated to g;
- K and h are given smooth functions on Σ, $\partial\Sigma$, respectively.

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Problem $(P_{K,h})$ is equivalent to the following geometric problem:

Prescribed curvatures problem

Is there a **conformal metric** $\widetilde{g} = e^v g$ whose Gaussian and geodesic curvatures are respectively $\widetilde{K}_g = K$ and $\widetilde{h}_g = h$?

If Σ is closed, namely $\partial\Sigma = \emptyset$, $(P_{K,h})$ is reduced to the very-well known Liouville-type PDE

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On the other hand, there are only few results concerning $(P_{K,h})$ in the general case:

- (Chang-Yang '88) when $h \equiv 0$;
- (Chang-Liu '96), (Li-Liu '05), (Liu-Huang '05) when $K \equiv 0$;
- (Brendle '02) in the case $K \equiv K_0$, $h \equiv h_0$ via parabolic flow;
- (Cruz, Ruiz '18) on $\Sigma = \mathbb{D}$ under symmetry assumptions;
- (López-Soriano, Malchiodi, Ruiz) under assumptions on K, h.

Mean-field approach

Problem (P_K) has an equivalent **mean-field** formulation

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$$v \text{ solves } (P_K) \quad \Rightarrow \quad v \text{ solves } (MF_\rho) \text{ with } \rho = \int_\Sigma Ke^u;$$

$$u \text{ solves } (MF_\rho) \quad \Rightarrow \quad u + \log \frac{\rho}{\int_\Sigma Ke^u} \text{ solves } (P_K).$$

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Mean field problem (MF_ρ) has the advantage of being **variational** on $H^1(\Sigma)$; with the energy functional being

$$\mathcal{J}_\rho(u) := \frac{1}{2}\int_\Sigma |\nabla u|^2 + 2\int_\Sigma K_g u - 2\rho \log \left| \int_\Sigma Ke^u \right|,$$

which can be handled using Moser-Trudinger type inequalities.

Mean-field approach

We then introduce a **double mean-field formulation** for $(P_{K,h})$:

$$\begin{cases} -\Delta u + 2K_g = 2\rho \dfrac{Ke^u}{\int_\Sigma Ke^u} & \text{in } \Sigma \\ \partial_\nu u + 2h_g = 2\rho' \dfrac{he^{\frac{u}{2}}}{\int_{\partial\Sigma} he^{\frac{u}{2}}} & \text{on } \partial\Sigma \end{cases} ; \qquad (MF_{\rho,\rho'})$$

it has the similar energy functional

$$\begin{aligned} \mathcal{J}_{\rho,\rho'}(u) &:= \frac{1}{2}\int_\Sigma |\nabla u|^2 + 2\int_\Sigma K_g u - 2\rho \log\left|\int_\Sigma Ke^u\right| \\ &+ 2\int_{\partial\Sigma} h_g u - 4\rho' \log\left|\int_{\partial\Sigma} he^{\frac{u}{2}}\right|. \end{aligned}$$

Mean-field approach

However, problems $(P_{K,h})$ and $(MF_{\rho,\rho'})$ are **not** equivalent:

$$v \text{ solves } (P_{K,h}) \quad \Rightarrow \quad v \text{ solves } (MF_{\rho,\rho'}),\ \rho = \int_\Sigma Ke^u, \rho' = \int_{\partial\Sigma} he^{\frac{u}{2}};$$

$$u \text{ solves } (MF_{\rho,\rho'}) \quad \Rightarrow \quad u + \log\frac{\rho}{\int_\Sigma Ke^u} \text{ solves } (P_{K,\mathbf{c}h})$$

$$\text{with} \quad \mathbf{c} = \sqrt{\frac{\int_\Sigma Ke^u}{\rho}}\frac{\rho'}{\int_{\partial\Sigma} he^{\frac{u}{2}}}.$$

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$$\text{with} \quad \mathbf{c} = \sqrt{\frac{\int_\Sigma Ke^u}{\rho}} \frac{\rho'}{\int_{\partial\Sigma} he^{\frac{u}{2}}}.$$

Such an issue has been tackled by (Cruz-Ruiz '18) as follows.
By the Gauss-Bonnet theorem,

$$\rho + \rho' = \int_\Sigma Ke^u + \int_{\partial\Sigma} he^{\frac{u}{2}} = \int_\Sigma K_g + \int_{\partial\Sigma} h_g = 2\pi\chi(\Sigma);$$

therefore, unlike the case $\partial\Sigma = \emptyset$, ρ is not prescribed.

Mean-field approach

We may look for solutions to $(MF_{\rho,\rho'})$ with ρ such that $\mathbf{c} = 1$, i.e.:

$$\begin{cases} -\Delta u + 2K_g = 2\rho \dfrac{Ke^u}{\int_\Sigma Ke^u} & \text{in } \Sigma \\ \partial_\nu u + 2h_g = 2(2\pi\chi(\Sigma) - \rho)\dfrac{he^{\frac{u}{2}}}{\int_{\partial\Sigma} he^{\frac{u}{2}}} & \text{on } \partial\Sigma \\ \dfrac{(2\pi\chi(\Sigma) - \rho)^2}{|\rho|} = \dfrac{\left(\int_{\partial\Sigma} he^{\frac{u}{2}}\right)^2}{\int_\Sigma Ke^u} \end{cases}.$$

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We still have a convenient variational formulation with

$$\begin{aligned} \mathcal{I}(u,\rho) &:= \frac{1}{2}\int_\Sigma |\nabla u|^2 + 2\int_\Sigma K_g u - 2\rho \log\left|\int_\Sigma Ke^u\right| \\ &- 4(2\chi(\Sigma) - \rho)\log\left|\int_{\partial\Sigma} he^{\frac{u}{2}}\right| + 2\int_{\partial\Sigma} h_g u + \mathcal{F}(\rho) \\ &= \mathcal{J}_{\rho,2\pi\chi(\Sigma)-\rho}(u) + \mathcal{F}(\rho). \end{aligned}$$

Mean-field approach

In (Cruz-Ruiz '18) solutions are found studying $\mathcal{J}_{\rho,2\pi\chi(\Sigma)-\rho}(u)$ and then the behavior of critical points u_ρ on varying ρ.

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Therefore, we will study $(MF_{\rho,\rho'})$ with **generic** K_g, h_g, ρ, ρ'.
It will not be restrictive to take $h_g \equiv 0$ and $K_g \equiv \dfrac{\rho+\rho'}{|\Sigma|}$, namely

$$\begin{cases} -\Delta u + \dfrac{2(\rho+\rho')}{|\Sigma|} = 2\rho \dfrac{Ke^u}{\int_\Sigma Ke^u} & \text{in } \Sigma \\ \partial_\nu u = 2\rho' \dfrac{he^{\frac{u}{2}}}{\int_{\partial\Sigma} he^{\frac{u}{2}}} & \text{on } \partial\Sigma \end{cases}.$$

We will only consider **constantly-signed** K, h with

$$\mathrm{sgn}(K) = \mathrm{sgn}(\rho) \qquad\qquad \mathrm{sgn}(h) = \mathrm{sgn}(\rho').$$

Blow-up analysis

Blow-up phenomena for problem $(MF_{\rho,\rho'})$ are similar to the ones for standard Liouville equation, though with some differences.

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Bao, Wang, Zhou '05; Lopez-Soriano, Malchiodi, Ruiz; B., L.-S.

Let $\{u_n\}$ be a sequence of solutions to (MF_{ρ_n,ρ'_n}).
Then, up to constants and to sub-sequences:

- *Either* $\{u_n\}_{n\in\mathbb{N}}$ is compact in $H^1(\Sigma)$;
- *Or* There exists a **finite** blow-up set $\mathcal{S} \neq \emptyset$ such that

$$\rho_n \frac{K_n e^{u_n}}{\int_\Sigma K_n e^{u_n}} \underset{n\to+\infty}{\rightharpoonup} 4\pi \sum_{p\in\mathcal{S}\cap\mathring{\Sigma}} \delta_p + \sum_{p\in\mathcal{S}\cap\partial\Sigma} \alpha_p \delta_p$$

$$\rho'_n \frac{h_n e^{\frac{u_n}{2}}}{\int_\Sigma h_n e^{\frac{u_n}{2}}} \underset{n\to+\infty}{\rightharpoonup} \sum_{p\in\mathcal{S}\cap\partial\Sigma} (2\pi - \alpha_p)\delta_p + \mu,$$

with $\alpha_p \in \mathbb{R}$, $\mu \in L^1(\partial\Sigma)$ and $\mu \equiv 0$ if $\mathcal{S}\cap\partial\Sigma \neq \emptyset$.

Blow-up analysis

The blow-up at $p \in \mathcal{S} \cap \mathring{\Sigma}$ is essentially the same as the standard Liouville equation, the limiting profile being

$$U(x) = \log \frac{4\lambda^2}{(1+\lambda^2|x|^2)^2} \qquad \begin{cases} -\Delta U = 2e^U & \text{in } \mathbb{R}^2 \\ \int_{\mathbb{R}^2} e^U < +\infty \end{cases} ;$$

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In case of blow up at $p \in \mathcal{S} \cap \partial\Sigma$, the limiting profile solves

$$\begin{cases} -\Delta U = 2ae^U & \text{in } \mathbb{R}^2_+ \\ \partial_\nu U = 2ce^{\frac{U}{2}} & \text{in } \partial\mathbb{R}^2_+ \\ \int_{\mathbb{R}^2_+} e^U + \int_{\partial\mathbb{R}^2_+} e^{\frac{U}{2}} < +\infty \end{cases} .$$

Blow-up analysis

Such entire solutions have been classified by (Zhang '03).
Depending on $\mathrm{sgn}(K(p))$, we have:

$$a = 1, c \in \mathbb{R} \quad \Rightarrow \quad U(x) = \log \frac{4\lambda^2}{\left(1 + \lambda^2 \left|x + \left(0, \frac{c}{\lambda}\right)\right|^2\right)^2};$$

$$a = 0, c > 0 \quad \Rightarrow \quad U(x) = 2 \log \frac{2}{\lambda \left|x + \left(0, \frac{c}{\lambda}\right)\right|^2};$$

$$a = -1, c > 1 \quad \Rightarrow \quad U(x) = \log \frac{4\lambda^2}{\left(\lambda^2 \left|x + \left(0, \frac{c}{\lambda}\right)\right|^2 - 1\right)^2}.$$

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In all cases, the sum of the local masses is $\int_{\mathbb{R}^2_+} e^U + \int_{\partial\mathbb{R}^2_+} e^{\frac{U}{2}} = 2\pi$.

Blow-up analysis

Therefore, if $\mathcal{S} \cap \partial\Sigma = \emptyset$, then $\rho = 4\pi M$ for some $M \in \mathbb{N}$.
On the other hand, if $\mathcal{S} \cap \partial\Sigma \neq \emptyset$, then $\rho + \rho' = 2\pi N$ for $N \in \mathbb{N}$.

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Conversely, blow-up cannot occur if $(\rho, \rho') \notin \Gamma$:

$$\Gamma := \{(\rho, \rho') \in \mathbb{R}^2 : \ \rho \in 4\pi\mathbb{N} \text{ or } \rho + \rho' \in 2\pi\mathbb{N}\}.$$

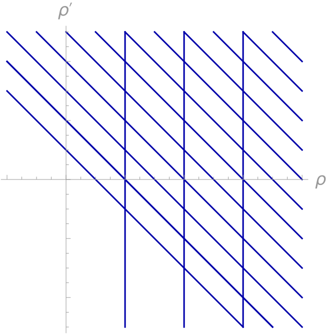

Figure: The set Γ of non-compactness values for (ρ, ρ').

Moser-Trudinger inequality

We look for solutions to $(MF_{\rho,\rho'})$ as critical points of

$$\mathcal{J}_{\rho,\rho'}(u) = \frac{1}{2}\int_\Sigma |\nabla u|^2 + \frac{2(\rho+\rho')}{|\Sigma|}\int_\Sigma u - 2\rho \log\left|\int_\Sigma K e^u\right| - 4\rho' \log\left|\int_{\partial\Sigma} h e^{\frac{u}{2}}\right|.$$

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Original Moser-Trudinger's inequality on closed surfaces reads as

Trudinger '68; Moser '71

$$8\pi \log \int_\Sigma e^u - \frac{8\pi}{|\Sigma|}\int_\Sigma u \le \frac{1}{2}\int_\Sigma |\nabla u|^2 + C, \qquad \forall\, u \in H^1(\Sigma).$$

Moser-Trudinger inequality

On surfaces with boundary $\partial\Sigma \neq \emptyset$ we get

Chang, Yang '88

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Li, Liu '05

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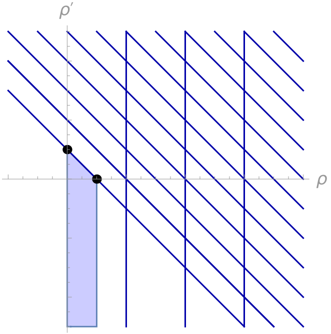
By interpolating the inequalities we get, if $\rho, \rho' \geq 0, \rho + \rho' \leq 2\pi$,

$$2\rho \log \int_\Sigma e^u + 4\rho' \log \int_{\partial\Sigma} e^{\frac{u}{2}} - \frac{2(\rho+\rho')}{|\Sigma|}\int_\Sigma u \leq \frac{1}{2}\int_\Sigma |\nabla u|^2 + C.$$

Moser-Trudinger inequality

Therefore $\mathcal{J}_{\rho,\rho'}$ is: bounded from below if $\rho, \rho' \geq 0, \rho + \rho' \leq 2\pi$;
coercive if $\rho, \rho' \geq 0, \rho + \rho' < 2\pi$.

In particular, in the latter case there exist minimizers to $\mathcal{J}_{\rho,\rho'}$.



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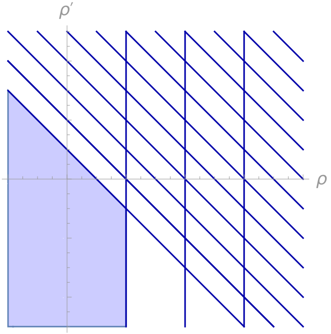

We can improve the result to get coercivity for $\rho < 4\pi, \rho + \rho' < 2\pi$.

Arguing as (Jost, Wang '01) for Liouville systems, we apply blow-up analysis to minimizers: if $\rho < 4\pi, \rho + \rho' < 2\pi$, blow-up is excluded, hence coercivity holds.

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Using test functions we also see that

$\mathcal{J}_{\rho,\rho'}$ is:	not bounded from below	if $\rho > 4\pi$ or $\rho + \rho' > 2\pi$;
	not coercive	if $\rho \geq 4\pi$ or $\rho + \rho' \geq 2\pi$.

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$\mathcal{J}_{\rho,\rho'}$ may still be bounded from below if $\rho = 4\pi$ or $\rho + \rho' = 2\pi$. To see this, we need a sharper blow-up analysis of minimizers $u_n = u_{\rho_n,\rho'_n}$ as $\rho_n + \rho'_n \nearrow 2\pi$ or $\rho_n \nearrow 4\pi$.

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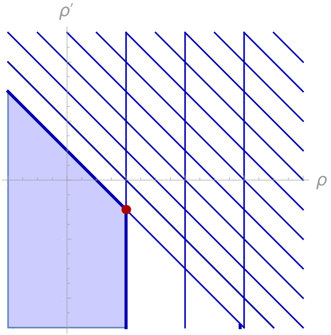
In view of the limiting profiles, we are able to show boundedness from below in all cases except $(4\pi, -2\pi)$:

Moser-Trudinger inequality



B., L.-S.

If $\rho \leq 4\pi, \rho + \rho' < 2\pi$ or $\rho < 4\pi, \rho + \rho' \leq 2\pi$, then

$$2\rho \log \int_\Sigma e^u + 4\rho' \log \int_{\partial\Sigma} e^{\frac{u}{2}} - \frac{2(\rho+\rho')}{|\Sigma|} \int_\Sigma u \leq \frac{1}{2} \int_\Sigma |\nabla u|^2 + C.$$

Min-max solutions

If $\rho < 4\pi$, $\rho + \rho' < 2\pi$ coercivity yields minimizing solutions, but for higher values we have to look for **min-max** solutions.

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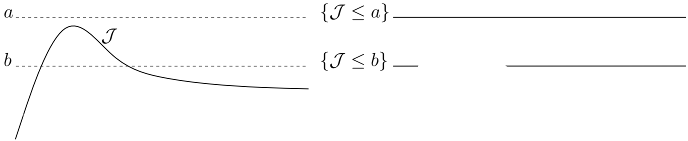
We get solutions from a change in the **topology of sublevels**.



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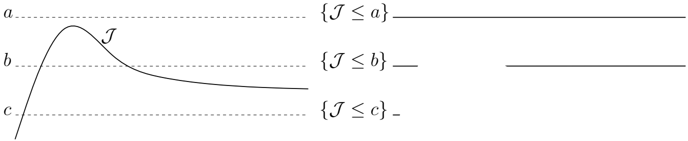

Non-compactness is excluded by assuming $(\rho, \rho') \notin \Gamma$, i.e.

$$4M\pi < \rho < 4(M+1)\pi, \qquad 2N\pi < \rho+\rho' < 2(N+1)\pi, \qquad M, N \in \mathbb{N}.$$

Min-max solutions

From compactness we get $\{\mathcal{J}_{\rho,\rho'} \leq L\}$ is contractible for $L \gg 0$;
We need to show that $\{\mathcal{J}_{\rho,\rho'} \leq -L\}$ is **not** contractible for $L \gg 0$.

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This will follow by finding a non-contractible $\mathcal{X}$ and maps

$$\mathcal{X} \overset{\Phi}{\to} \left\{\mathcal{J}_{\rho,\rho'} \leq -L\right\} \overset{\Psi}{\to} \mathcal{X} \qquad \text{such that} \qquad \Psi \circ \Phi \simeq \mathrm{Id}_{\mathcal{X}}.$$

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To construct Ψ, Φ, we see that if $J_{\rho,\rho'}(u) \ll 0$, then $\dfrac{Ke^u}{\int_\Sigma Ke^u}$ concentrates at a finite number of points depending on ρ, ρ'.

Min-max solutions

Barycenters are a model for concentration at finitely many points:

$$(\Omega)_K := \left\{ \sum_{i=1}^{K} t_i \delta_{p_i}, \; \sum_{i=1}^{K} t_i = 1, \; p_i \in \Omega \right\}.$$

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$$(\Omega)_K := \left\{ \sum_{i=1}^{K} t_i \delta_{p_i},\ \sum_{i=1}^{K} t_i = 1,\ p_i \in \Omega \right\}.$$

In particular, we can construct maps Ψ, Φ using

$$\mathcal{X} := \begin{cases} \left(\widetilde{\Sigma}\right)_M & M \geq N \\ (\partial\Sigma)_N & M < N \end{cases}, \qquad \text{for some deformation retract} \quad \widetilde{\Sigma} \Subset \Sigma.$$

Min-max solutions

Barycenters are a model for concentration at finitely many points:

$$(\Omega)_K := \left\{ \sum_{i=1}^{K} t_i \delta_{p_i},\ \sum_{i=1}^{K} t_i = 1,\ p_i \in \Omega \right\}.$$

In particular, we can construct maps Ψ, Φ using

$$\mathcal{X} := \begin{cases} \left(\widetilde{\Sigma}\right)_M & M \geq N \\ (\partial\Sigma)_N & M < N \end{cases}, \qquad \text{for some deformation retract} \quad \widetilde{\Sigma} \Subset \Sigma.$$

We need to verify whether $\mathcal{X}$ is contractible:

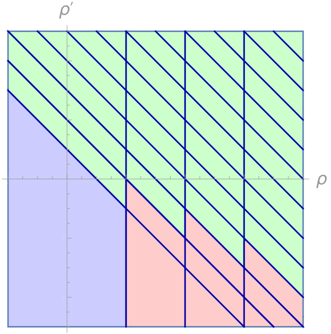
If $M \geq N$, $\left(\widetilde{\Sigma}\right)_M$ is contractible $\iff$ Σ is simply connected;
If $M < N$, $(\partial\Sigma)_N$ is always non-contractible.

Min-max solutions

Therefore we get:

B., L.-S.

Assume $4M\pi < \rho < 4(M+1)\pi, 2N\pi < \rho + \rho' < 2(N+1)\pi$.
If Σ is simply connected, then $(MF_{\rho,\rho'})$ has solutions for $M < N$.
If Σ is multiply connected, then $(MF_{\rho,\rho'})$ has solutions for all M, N.



THANK YOU FOR YOUR ATTENTION!