Nonlinear Geometric PDE's

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1 Overview of the Field

The study of geometric PDEs has been fundamental to solve very classical problems in geometry, and has given rise to new challenging ones. This area of mathematics mixes together different ideas and tools coming from geometry and from analysis. On one side, the geometric structure of these PDEs often translates into technical difficulties related to the presence of some intrinsic geometric invariance of the problem, like conformal invariance, gauge invariance, etc, that is reflected in a possible lack of compactness of the functional embeddings for the natural spaces of functions associated with the problems. Technical tools coming from analysis are often crucial to overcome this type of difficulties, among others related to regularity or a priori estimates on solutions. On the other hand, the geometric intuition of the problem always contributes to the identification for the natural quantities to keep track of, and suggests the correct result to pursue. This two-fold aspect of the study of geometric PDEs makes it both challenging and complex, and require the use of several refined techniques to overcome the major difficulties encountered. These are the main reasons why Geometric PDEs are a field of research which is currently very active.

The use of nonlinear PDEs arguments in geometric problems has progressed very rapidly in recent years. Thus it is timely to have a new BIRS workshop on Nonlinear Geometric PDEs which covers different topics of the field. Actually, a periodic workshop on these themes would be expected after a couple of years, to see much further developments, with a lot of interesting new results and directions to discuss.

2 Outcome of the meeting

The workshop gathered international researchers in the areas of geometric analysis and geometric and non-linear partial differential equations (PDEs).

In particular the following topics have been addressed.

1. Local and non-local elliptic equations from conformal geometry.

The general aim is to deform the metric of a given manifold in a conformal way so that the new one possesses special properties. Classical examples are the uniformization problem of two dimensional surfaces and the Yamabe problem in dimension greater or equal to three. In particular, the problem of prescribing the Q-curvature discovered by Branson has been intensely studied by analysts as a generalization of scalar curvature in Yamabe-type problems. The sigma(k)-Yamabe problem is a generalization of the Yamabe problem introduced by Viaclovsky and it has spawned vast activity and progress in the analysis of fully nonlinear partial differential equations.

The classical example of the Yamabe problem in dimension greater or equal to three has a variational structure and it can be expressed in the form of a non-linear PDEs on the ambient manifold for the conformal factor. Existence and qualitative properties of positive solutions to such equations are by now considerably well understood. Much less is known on the existence of sign-changing solutions.

Some of the results presented in the conference are:

- **Juan Carlos Fernández** *Supercritical problems on the round sphere and the Yamabe problem in projective spaces*. Given an isoparametric function f on the round sphere and considering the space of functions $w \circ f$, the Yamabe-type problem can be reduced to

$$(1) \qquad -\Delta_{g_0} + \lambda u = \lambda |u|^{p-1} u \text{ on } S^n$$

with $\lambda > 0$ and $p > 1$, into a second order singular ODE of the form

$$w'' + \frac{h(r)}{\sin r} w' + \lambda \left(|w|^{p-1} w - w \right) = 0,$$

with boundary conditions $w'(0) = 0$ and $w'(\pi) = 0$, and where h is a monotone function with exactly one zero on $[0, \pi]$. Using a double shooting method, for any $k \in N$, if $n_1 \leq n_2$ are the dimensions of the focal submanifolds determined by f and if $p \in \left(1, \frac{n-n_1+2}{n-n_1-2}\right)$, this problem admits a nodal solution having at least k zeroes. This yields a solution to problem (1) having as nodal set a disjoint union of at least k connected isoparametric hypersurfaces. As an application and using that the Hopf fibrations are Riemannian submersions with minimal fibers, we give a multiplicity result of nodal solutions to the Yamabe problem on CP^m and on HP^m, the complex and quaternionic projective spaces respectively, with m odd.

- **Bruno Premoselli** *Compactness of sign-changing solutions to scalar curvature-type equations with bounded negative part.*
Given the equation $\triangle_g u + hu = |u|^{2^*-2} u$ in a closed Riemannian manifold (M, g), where h is some Holder continuous function in M and $2^* = \frac{2n}{n-2}$, $n := dim M$, he is interested in a sharp compactness result on the sets of sign-changing solutions whose negative part is a priori bounded. The result is obtained under the conditions that $n \geq 7$ and $h < (n-2)/(4(n-1)) S_g$ in M, where S_g is the Scalar curvature of the manifold. These conditions are optimal by constructing examples of blowing-up solutions, with arbitrarily large energy, in the case of the round sphere with a constant potential function h.

- **Jérôme Vétois** *Influence of the scalar curvature and the mass on blowing-up solutions to low-dimensional conformally invariant equations.*
A result of Olivier Druet provides necessary conditions for the existence of blowing-up solutions to a class of conformally invariant elliptic equations of second order on a closed Riemannian manifold whose energy is a priori bounded. Essentially, these conditions say that for such solutions to exist, the potential function in the limit equation must coincide, up to a constant factor, at least at one point, with the scalar curvature of the manifold and moreover, in low dimensions, the weak limit of the solutions must be identically zero. New existence results show the optimality of these conditions, in particular in the case of low dimensions.

A new recent notion of fractional curvature leads to a non-local Yamabe problem, namely the existence of a metric, conformally equivalent to the given one, whose fractional scalar curvature is constant.

- **Maria del Mar Gonzalez** *Nonlocal ODE, conformal geometry and applications.*
We study radially symmetric solutions for a semilinear equation with fractional Laplacian. Contrary to the local case, where we can give a solution to an ODE by simply looking at its phase portrait, in the

nonlocal case we develop several new methods. We will give some applications, in particular to the existence of solutions of the singular fractional Yamabe problem, and the uniqueness of steady states of aggregation-diffusion equations.

- **Seunghyeok Kim** *A compactness theorem of the fractional Yamabe problem.*
  Since Schoen raised the question of compactness of the full set of solutions of the Yamabe problem in the C^0 topology (in 1988), it had been generally expected that the solution set must be C^0-compact unless the underlying manifold is conformally equivalent to the standard sphere. In 2008-09, Khuri, Marques, Schoen himself and Brendle gave the surprising answer that the expectation holds whenever the dimension of the manifold is less than 25 (under the validity of the positive mass theorem whose proof is recently announced by Schoen and Yau) but does not if the dimension is 25 or greater. On the other hand, concerning the fractional Yamabe problem on a conformal infinity of an asymptotically hyperbolic manifold, Kim, Musso, and Wei considered an analogous question and constructed manifolds of high dimensions whose solution sets are C^0-noncompact (in 2017). In this talk, we show that the solution set is C^0-compact if the conformal infinity is non-umbilic and its dimension is 7 or greater. Our proof provides a general scheme toward other possible compactness theorems for the fractional Yamabe problem.

Some related problems has been studied by

- **Gabriella Tarantello** *Minimal immersions of closed surfaces in hyperbolic 3- manifold.*
  Motivated by the the work of K. Uhlenbeck, we discuss minimal immersions of closed surfaces of genus larger than 1 on hyperbolic 3-manifold. In this respect we establish multiple existence for the Gauss-Codazzi equations and describe the asymptotic behaviour of the solutions in terms of the prescribed conformal structure and holomorphic quadratic differential whose real part identifies the corresponding second fundamental form.

- **Pierpaolo Esposito** *Log-determinants in conformal geometry.*
  I will report on a recent result concerning a four-dimensional PDE of Liouville type arising in the theory of log-determinants in conformal geometry. The differential operator combines a linear fourth-order part with a quasi-linear second-order one. Since both have the same scaling behavior, compactness issues are very delicate and even the "linear theory" is problematic. For the log-determinant of the conformal laplacian and of the spin laplacian we succeed to show existence and logarithmic behavior of fundamental solutions, quantization property for non-compact solutions and existence results via critical point theory.

- **Andrea Malchiodi** *On the Sobolev quotient in sub-Riemannian geometry.*
  We consider a class of three-dimensional "CR manifolds" which are modelled on the Heisenberg group. We introduce a natural concept of "mass" and prove its positivity under the conditions that the Webster curvature is positive and in relation to their (holomorphic) embeddability properties. We apply this result to the CR Yamabe problem, and we discuss the properties of Sobolev-type quotients, giving some counterexamples to the existence of minimisers for "Rossi spheres", in sharp contrast to the Riemannian case.

2. Properties of solutions to PDE's on manifolds.

The goal is to investigate some properties, such as rigidity, regularity, stability, of solutions of nonlinear PDEs on manifolds.

- **Virginia Agostiniani** *Monotonicity formulas in linear and nonlinear potential theory.*
  In this talk, we first recall how some monotonicity formulas can be derived along the level set flow of the capacitary potential associated with a given bounded domain Ω. A careful analysis is required in order to preserve the monotonicity across the singular times, leading in turn to a new quantitative version of the Willmore inequality. Remarkably, such analysis can be carried out without any *a priori* knowledge of the size of the singular set. Hence, the same order of ideas applies to the p-capacitary potential of Ω, whose critical set, for $p \neq 2$, is not necessarily negligible. In this context, a generalised version of the Minkowski inequality is deduced.

- **Giovanni Catino** *Some canonical Riemannian metrics: rigidity and existence.*
In this talk, which is the second part of a joint seminar with P. Mastrolia (Universit degli Studi di Milano), I will present some results concerning rigidity and existence of canonical metrics on closed (compact without boundary) four manifolds. In particular I will consider Einstein metrics, Harmonic Weyl metrics and some generalizations.

- **Paolo Mastrolia** *Generalizations of some canonical Riemannian metrics.*
In this talk, which is the first part of a joint seminar with G. Catino (Politecnico di Milano), I will introduce some generalization of certain canonical Riemannian metrics, presenting two possible approaches (curvature conditions with potential and critical metrics of Riemannian functionals). The main result is related to the existence of a new canonical metric, which generalizes the condition of harmonic Weyl curvature, on every 4-dimensional closed manifold.

- **Lorenzo Mazzieri** *Sharp Geometric Inequalities on manifolds with nonnegative Ricci curvature.*
Given a complete Riemannian manifold with nonnegative Ricci curvature and Euclidean volume growth, we characterize the Asymptotic Volume Ratio as the infimum of the Willmore Energy over smooth closed hypersurfaces. An optimal version of Huisken's Isoperimetric Inequality for 3-manifolds is obtained as a consequence of this result.

- **Dario Monticelli** *The Poisson equation on Riemannian manifolds with a weighted Poincaré inequality at infinity.*
We prove an existence result for the Poisson equation on non-compact Riemannian manifolds satisfying a weighted Poincaré inequality outside a compact set. Our result applies to a large class of manifolds including, for instance, all non-parabolic manifolds with minimal positive Green's function vanishing at infinity. On the source function we assume a sharp pointwise decay depending on the weight appearing in the Poincaré inequality and on the behavior of the Ricci curvature at infinity. We do not require any curvature or spectral assumptions on the manifold.

- **Roger Moser** *On a type of second order variational problem in L-infinity.*
Let K be an elliptic (not necessarily linear) second order differential operator. Suppose that we want to minimise the L-infinity norm of K(u) for functions u satisfying suitable boundary conditions. Here K may represent, e.g., the curvature of a curve in the plane or the scalar curvature of a Riemannian manifold in a fixed conformal class, but the problem is not restricted to questions with a geometric background. If the operator and the boundary conditions are such that the equation K(u) = 0 has a solution, then the problem is of course trivial. But since this is a second order variational problem, it may be appropriate to prescribe u as well as its first derivative on the boundary of its domain, which in general rules out this situation. In the cases studied so far, the solution, while not trivial, still has a nice structure, and one feature is that —K(u)— is always constant. The sign of K(u) may jump, but we have a characterisation of the jump set in terms of a linear PDE. Furthermore, in some cases we have a unique solution, even though the underlying functional is not strictly convex.

- **Susanna Terracini** *Liouville type theorems and local behaviour of solutions to degenerate or singular problems.*
We consider an equation in divergence form with a singular/degenerate weight
$$-\mathrm{div}(|y|^a A(x,y)\nabla u) = |y|^a f(x,y) \quad \text{or } \mathrm{div}(|y|^a F(x,y)) \ ,$$
Under suitable regularity assumptions for the matrix A and f (resp. F) we prove Hölder continuity of solutions and possibly of their derivatives up to order two or more (Schauder estimates). In addition, we show stability of the $C^{0,\alpha}$ and $C^{1,\alpha}$ a priori bounds for approximating problems in the form
$$-\mathrm{div}((\varepsilon^2+y^2)^a A(x,y)\nabla u) = (\varepsilon^2+y^2)^a f(x,y) \quad \text{or } \mathrm{div}((\varepsilon^2+y^2)^a F(x,y))$$
as $\varepsilon \to 0$. Finally, we derive $C^{0,\alpha}$ and $C^{1,\alpha}$ bounds for inhomogenous Neumann boundary problems as well. Our method is based upon blow-up and appropriate Liouville type theorems.

3. Geometric evolution equations

Many geometric variational problems can be approached via evolutionary methods. The Yamabe flow in conformal geometry and the mean curvature flow are two examples with connections to problems in differential geometry and mathematical physics.

Results on the analysis of possible blow-up phenomena for several non-linear flows have been presented.

- **Monica Musso** *Singularity formation in critical parabolic equations.*
  In this talk I will discuss some recent constructions of blow-up solutions for a Fujita type problem for powers p related to the critical Sobolev exponent. Both finite type blow-up (of type II) and infinite time blow-up are considered.

- **Mariel Saez** *On the uniqueness of graphical mean curvature flow.*
  In this talk I will discuss on sufficient conditions to prove uniqueness of complete graphs evolving by mean curvature flow. It is interesting to remark that the behaviour of solutions to mean curvature flow differs from the heat equation, where non-uniqueness may occur even for smooth initial conditions if the behaviour at infinity is not prescribed for all times.

- **Juan Davila** *Helicoidal vortex filaments in the 3-dimensional Ginzburg-Landau equation.*
  We construct a family of entire solutions of the 3D Ginzburg-Landau equation with vortex lines given by interacting helices, with degree one around each filament and total degree an arbitrary positive integer. Existence of these solutions was conjectured by del Pino and Kowalczyk (2008), and answers negatively a question of Brezis analogous to the the Gibbons conjecture for the Allen-Cahn equation.

- **Manuel del Pino** *Singularity for the Keller-Segel system in R^2.*
  We construct solutions of the Keller-Segel system which blow-up in infinite time in the form of asymptotic aggregation in the critical mass case, with a method that does not rely on radial symmetry, and applies to establish stability of the phenomenon.

4. Concentration phenomena in local and non-local problems.

The geometric invariances often lead to the existence of solutions which blows-up at single points or at higher dimensional sets. The localization of the blowing-up sets strongly depends on the geometric properties of the problem. It is an active field in nonlinear analysis to constructs solutions to PDEs exhibiting this kind of phenomena.

Problems in 2D have been treated in

- **Weiwei Ao** *On the bubbling solutions of the Maxwell-Chern-Simons model on flat torus*
  We consider the periodic solutions of a nonlinear elliptic system derived from the Maxwell-Chern-Simons model on a flat torus Ω:

  $$\begin{cases} \Delta u = \mu(\lambda e^u - N) + 4\pi \sum_{i=1}^n m_i \delta_{p_i}, \\ \Delta N = \mu(\mu + \lambda e^u)N - \lambda\mu(\lambda + \mu)e^u \end{cases} \text{in } \Omega,$$

  where $\lambda, \mu > 0$ are positive parameters. We obtain a Brezis-Merle type classification result for this system when $\lambda, \mu \to \infty$ and $\lambda << \mu$. We also construct blow up solutions to this system.

- **Luca Battaglia** *A double mean field approach for a curvature prescription problem.*
  I will consider a double mean field-type Liouville PDE on a compact surface with boundary, with a nonlinear Neumann condition. This equation is related to the problem of prescribing both the Gaussian curvature and the geodesic curvature on the boundary. I will discuss blow-up analysis, a sharp Moser-Trudinger inequality for the energy functional, existence of minmax solution when the energy functional is not coercive.

- **Teresa D'Aprile** *Non simple blow-up phenomena for the singular Liouville equation.*
  Let Ω be a smooth bounded domain in R^2 containing the origin. We are concerned with the following Liouville equation with Dirac mass measure

  $$-\Delta u = \lambda e^u - 4\pi N_\lambda \delta_0 \quad \text{in} \quad \Omega,$$

with $u = 0$ on $\partial\Omega$. Here λ is a positive small parameter, δ_0 denotes Dirac mass supported at 0 and N_λ is a positive number close to an integer N ($N \geq 2$) from the right side. We assume that Ω is $(N+1)$-symmetric and the regular part of the Green's function satisfies a nondegeneracy condition (both assumptions are verified if Ω is the unit ball) and we provide an example of non-simple blow-up as $\lambda \to 0^+$ exhibiting a non-symmetric scenario. More precisely we construct a family of solutions split in a combination of $N+1$ bubbles concentrating at 0 arranged on a tiny polygonal configuration centered at 0.

- **Massimo Grossi** *Non-uniqueness of blowing-up solutions to the Gelfand problem.*
I will consider blowing-up solution for the Gelfand problem on planar domains. It is well known that blow up at a single point must occur at a critical point x of a "reduced functional" F, whereas uniqueness of blowing up families has been recently shown provided x is a non-degenerate critical point of F. We showed that, if x is a degenerate critical point of F and satisfies some additional generic condition, then one may have two solutions blowing up at the same point. Solutions are constructed using a Lyapunov-Schmidt reduction.

- **Gabriele Mancini** *Bubbling nodal solutions for a large perturbation of the Moser-Trudinger equation on planar domains.*
I will discuss some results obtained in collaboration with Massimo Grossi, Angela Pistoia and Daisuke Naimen concerning the existence of nodal solutions for the problem $-\Delta u = \lambda u e^{u^2+|u|^p}$ in Ω, $u = 0$ on $\partial\Omega$, where $\Omega \subseteq R^2$ is a bounded smooth domain and $p \to 1^+$. If Ω is ball, it is known that the case $p = 1$ defines a critical threshold between the existence and the non-existence of radially symmetric sign-changing solutions with λ close to 0. In our work we construct a blowing-up family of nodal solutions to such problem as $p \to 1^+$, when Ω is an arbitrary domain and λ is small enough. To our knowledge this is the first construction of sign-changing solutions for a Moser-Trudinger type critical equation on a non-symmetric domain.

- **Luca Martinazzi** *Topological and variational methods for the supercritical Moser-Trudinger equation.*
We discuss the existence of critical points of the Moser-Trudinger functional in dimension 2 with arbitrarily prescribed Dirichlet energy using degree theory. If time permits, we will also sketch an approach on Riemann surfaces using a min-max method á la Djadli-Malchiodi.

- **Aleks Jevnikar** *Uniqueness and non-degeneracy of bubbling solutions for Liouville equations.*
We prove uniqueness and non-degeneracy of solutions for the mean field equation blowing-up on a non-degenerate blow-up set. Analogous results are derived for the Gelfand equation. The argument is based on sharp estimates for bubbling solutions and suitably defined Pohozaev-type identities.

A non-local version of these kind of problems has been studied in

- **Azahara DelaTorre** *The non-local mean-field equation on an interval.*
We study the quantization properties for a non-local mean-field equation and give a necessary and sufficient condition for the existence of solution for a "Mean Field"-type equation in an interval with Dirichlet-type boundary condition. We restrict the study to the 1-dimensional case and consider the fractional mean-field equation on the interval $I = (-1, 1)$

$$(-\Delta)^{\frac{1}{2}} u = \rho \frac{e^u}{\int_I e^u dx},$$

subject to Dirichlet boundary conditions. As in the $2-$dimensional case, it is expected that for a sequence of solutions to our equation, either we get a $\mathcal{C}^\infty$ limiting solution or, after a suitable rescaling, we obtain convergence to the Liouville equation. Then, we can reduce the problem to the study of the non-local Liouville's equation. One of the key points here is to find an appropriate Pohozaev identity. We prove that existence holds if and only if $\rho < 2\pi$. This requires the study of blowing-up sequences of solutions. In particular, we provide a series of tools which can be used (and extended) to higher-order mean field equations of non-local type. We provide a completely non-local method for this study, since we do not use the localization through the extension method. Instead, we use the study of blowing-up sequences of solutions.

A related problem

- **Michal Kowalczyk** *New multiple end solutions in the Allen-Cahn and the generalized second Painlevé equation.*
In this talk I will discuss two new constructions of the multiple end solutions. In the case of the Allen-Cahn equation the ends are asymptotic to the Simons cone in R^8. The case of the generalized second Painlevé equation in R^2 is somehow different since there is no apparent underlying geometric problem. Yet we can interpret the behavior of the solution as being asymptotic along the axis to: two one dimensional Hastings-McLeod solution, the heteroclinic solution of the Allen-Cahn equation and the trivial solution.

- **Bob Jerrard** *Some Ginzburg-Landau problems for vector fields on manifolds.*
Motivated in part by problems arising in micromagnetics, we study several variational models of Ginzburg-Landau type, depending on a small parameter $\epsilon > 0$, for (tangent) vector fields on a 2-dimensional compact Riemannian surface. As $\epsilon \to 0$, the vector fields tend to be of unit length and develop singular points of a (non-zero) index, called vortices. Our main result determines the interaction energy between these vortices as $\epsilon \to 0$, allowing us to characterize the asymptotic behaviour of minimizing sequence.

Critical problems in higher dimension where a concentration phenomena naturally appears.

- **Thomas Bartsch** *A spinorial analogue of the Brezis-Nirenberg theorem.*
Let (M, g, σ) be a compact Riemannian spin manifold of dimension $m \geq 2$, let $S(M)$ denote the spinor bundle on M, and let D be the Atiyah-Singer Dirac operator acting on spinors $\Psi : M \to S(M)$. We present recent results on the existence of solutions of the nonlinear Dirac equation with critical exponent
$$D\Psi = \lambda\Psi + f(|\Psi|)\Psi + |\Psi|^{\frac{2}{m-1}}\Psi$$
where $\lambda \in R$ and $f(|\Psi|)\Psi$ is a subcritical nonlinearity in the sense that $f(s) = o\left(s^{\frac{2}{m-1}}\right)$ as $s \to \infty$.

- **Riccardo Molle** *Nonexistence results for elliptic problems in contractible domains.*
In this talk I will consider nonlinear elliptic equations involving the Laplace or the p-Laplace operator and nonlinearities with supercritical growth, from the viewpoint of the Sobolev embedding. I'll present some new nonexistence results in contractible and non starshaped domains. The domains that are considered can be arbitrarily close to non contractible domains and their geometry can be very complex.

- **Frédéric Robert** *Hardy-Sobolev critical equation with boundary singularity: multiplicity and stability of the Pohozaev obstruction.*
Let Ω be a smooth bounded domain in R^n $(n \geq 3)$ such that $0 \in \partial\Omega$. In this talk, we consider issues of non-existence, existence, and multiplicity of variational solutions for the borderline Dirichlet problem,
$$\begin{cases} -\Delta u - \gamma\frac{u}{|x|^2} - h(x)u = \frac{|u|^{2^\star(s)-2}u}{|x|^s} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{E}$$
where $0 < s < 2$, $2^\star(s) := \frac{2(n-s)}{n-2}$, $\gamma \in R$ and $h \in C^0(\overline{\Omega})$. We use sharp blow-up analysis on – possibly high energy– solutions of corresponding subcritical problems to establish, for example, that if $\gamma < \frac{n^2}{4} - 1$ and the principal curvatures of $\partial\Omega$ at 0 are non-positive but not all of them vanishing, then Equation (E) has an infinite number of (possibly sign-changing) solutions. This complements results of the first and third authors, who showed in that if $\gamma \leq \frac{n^2}{4} - \frac{1}{4}$ and the mean curvature of $\partial\Omega$ at 0 is negative, then (E) has a positive solution. On the other hand, our blow-up analysis also allows us to prove that if the mean curvature at 0 is positive, then there is a surprising stability of regimes where there are no variational positive solutions under C^1-perturbations of the potential h. In particular, we show non-existence of such solutions for (E) whenever Ω is star-shaped and h is close to 0, which include situations not covered by the classical Pohozaev obstruction.