De Finetti Theorems for Quantum Channels

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arXiv:1810.12197
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Banff 07/25/2019

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Outline

Motivation: Noisy Channel Coding

De Finetti Theorems

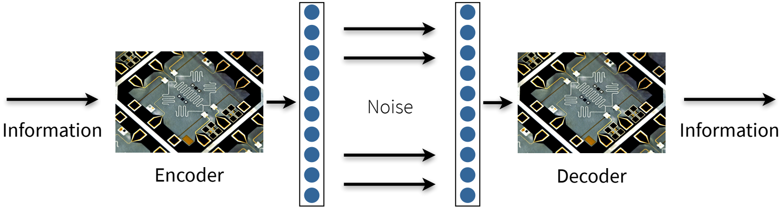
Application: Noisy Channel Coding

Conclusion

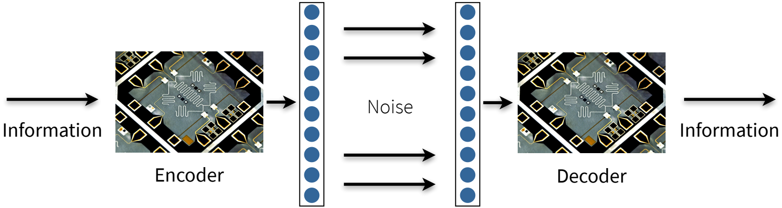
Add-on: De Finetti with Linear Constraints

Motivation: Noisy Channel Coding

Noisy Channel Coding



Noisy Channel Coding



Error Correction

m bits are subject to noise modelled by $N(y|x)$, find encoder e and decoder d to maximize probability $p(N, m)$ of retrieving m bits

Noisy Channel Coding (continued)

- Fixed number of bits m and noise model N gives **bilinear optimization**

$$
\begin{aligned}
p(N,m) = \max_{(e,d)} \quad & \frac{1}{2^m} \sum_{x,y,i} N(y|x)d(i|y)e(x|i) \\
\text{s.t.} \quad & \sum_{x} e(x|i) = 1,\ 0 \le e(x|i) \le 1 \\
& \sum_{i} d(i|y) = 1,\ 0 \le d(i|y) \le 1
\end{aligned}
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$$

- Approximating $p(N,m)$ up to multiplicative factor better than $(1 - e^{-1})$ is **NP-hard** in the worst case [Barman & Fawzi 18]

Noisy Channel Coding (continued)

- For the linear program [Hayashi 09, Polyanski *et al.* 10]

$$
\begin{aligned}
\mathrm{lp}(N,m) = \max_{(r,p)} \quad & \frac{1}{2^m}\sum_{x,y} N(y|x) r_{xy} \\
\text{s.t.} \quad & \sum_x r_{xy} \le 1, \ \sum_x p_x = k \\
& r_{xy} \le p_x, \ 0 \le r_{xy}, p_x \le 1
\end{aligned}
$$

  we have the approximation [Barman & Fawzi 18]

$$
p(N,m) \le \mathrm{lp}(N,m) \le \frac{1}{1-e^{-1}} \cdot p(N,m)
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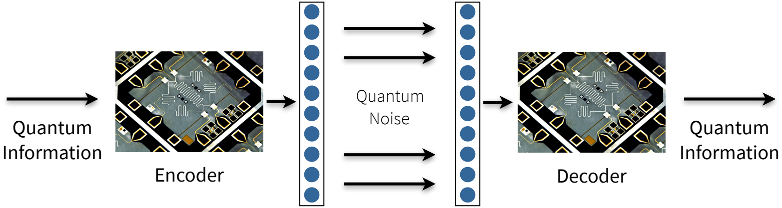
- **Polynomial** $(1-e^{-1})$-multiplicative approximation algorithms

Quantum Noisy Channel Coding

- Main question: Similar results for quantum error correction? [Matthews 12, Leung & Matthews 15]



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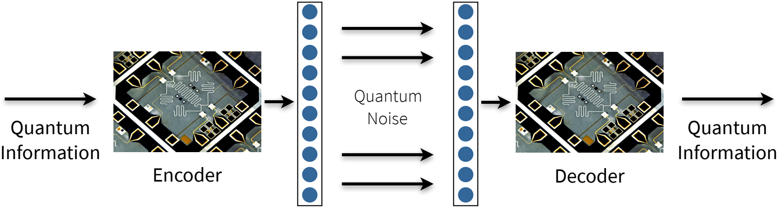
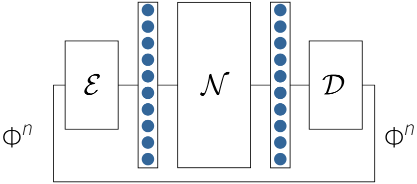


Quantum Error Correction

Find encoder E and decoder D to maximize quantum probability $F(\mathcal{N}, m)$ of retrieving m qubits

Quantum Noisy Channel Coding (continued)

- Near-term quantum devices are of intermediate scale and noisy



- Tailor-made **approximation algorithms** for encoder/ decoder?

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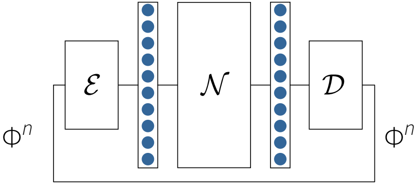

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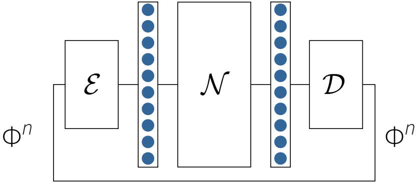
Optimize Quantum Information Processing

Develop mathematical toolbox rooted in optimization theory

Quantum Noisy Channel Coding (continued)



- m qubits with quantum noise model $\mathcal{N}$ leads to **quantum channel fidelity**

$$
\begin{aligned}
F(\mathcal{N}, n) := \max \quad & F\Big(\Phi^n, \big((\mathcal{D} \circ \mathcal{N} \circ \mathcal{E}) \otimes \mathcal{I}\big)(\Phi^n)\Big) \\
\text{s.t.} \quad & \mathcal{E}, \mathcal{D} \text{ quantum operations} \\
& \text{(+ physical constraints)}
\end{aligned}
$$

with fidelity $F(\rho, \sigma) := \|\sqrt{\rho}\sqrt{\sigma}\|_1^2$.

Quantum Noisy Channel Coding (continued)

- For $d := \dim(\mathcal{N})$ becomes **bilinear optimization**

$$F(\mathcal{N}, n) = \max \quad d \cdot \mathrm{Tr}\Big[\Big(\mathcal{N}_{\bar{A}\to B}(\Phi_{\bar{A}\bar{A}}) \otimes \Phi_{A\bar{B}}\Big)\Big(\sum_{i\in I} p_i \mathcal{E}^i_{A\to\bar{A}} \otimes \mathcal{D}^i_{B\to\bar{B}}\Big)(\Phi_{AA} \otimes \Phi_{BB})\Big]$$

$$\text{s.t.} \quad \mathcal{E}^i, \mathcal{D}^i \text{ quantum operations}, \; p_i \geq 0, \; \sum_{i\in I} p_i = 1$$

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- To characterize is set $\mathrm{SEP}_{\mathcal{N}}(A\bar{A}|B\bar{B})$ of **separable channels**

$$\sum_{i\in I} p_i \mathcal{E}^i_{A\to\bar{A}} \otimes \mathcal{D}^i_{B\to\bar{B}}$$

$\Rightarrow$ **strong hardness** for quantum separability [Barak *et al.* 12]

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- Lower bounds on figure of merit via, e.g., physical intuition or iterative see-saw methods $\Rightarrow$ **upper bounds**?

De Finetti Theorems

Monogamous Entanglement

- Quantum states ρ_{AB} is called k-**shareable** if

$$\rho_{AB_1\ldots B_k} \text{ with } \rho_{AB_j} = \rho_{AB} \ \forall j \in [k]$$

$\Rightarrow$ characterizes separable states [Stoermer 69, Doherty *et al.* 02]

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$\Rightarrow$ characterizes separable states [Stoermer 69, Doherty *et al.* 02]

De Finetti for Quantum States

For states $\rho_{AB_1^k} = \pi_{B_1^k}(\rho_{AB_1^k})$ we have that [Christandl *et al.* 07]

$$\min_{\{p_i, \sigma^i\}} \left\| \rho_{AB} - \sum_i p_i \sigma_A^i \otimes \sigma_B^i \right\|_1 \leq \frac{d_B^2}{k}$$

k-shareable Quantum Channels

De Finetti for Quantum Channels

For channels $\mathcal{N}_{AB_1^k \to \bar{A}\bar{B}_1^k}\left(\pi_{B_1^k}(\cdot)\right) = \pi_{\bar{B}_1^k}\left(\mathcal{N}_{AB_1^k \to \bar{A}\bar{B}_1^k}(\cdot)\right)$ with

$$\mathrm{Tr}_{\bar{A}}\left[\mathcal{N}_{AB_1^k \to \bar{A}\bar{B}_1^k}(\cdot)\right] = \mathrm{Tr}_{\bar{A}}\left[\mathcal{N}_{AB_1^k \to \bar{A}\bar{B}_1^k}\left(\frac{1_A}{d_A} \otimes \mathrm{Tr}_A[\cdot]\right)\right]$$

$$\mathrm{Tr}_{\bar{B}_k}\left[\mathcal{N}_{AB_1^k \to \bar{A}\bar{B}_1^k}(\cdot)\right] = \mathrm{Tr}_{\bar{B}_k}\left[\mathcal{N}_{AB_1^k \to \bar{A}\bar{B}_1^k}\left(\mathrm{Tr}_{B_k}[\cdot] \otimes \frac{1_{B_k}}{d_B}\right)\right]$$

we have that (cf. asymptotic bounds [Fuchs *et al.* 04])

$$\min_{\{p_i, \mathcal{E}^i, \mathcal{D}^i\}} \left\| \mathcal{N}_{AB \to \bar{A}\bar{B}} - \sum_{i \in I} p_i \mathcal{E}^i_{A \to \bar{A}} \otimes \mathcal{D}^i_{B \to \bar{B}} \right\|_{\diamond} \leq \sqrt{\frac{\mathrm{poly}(d_A d_{\bar{A}} d_B d_{\bar{B}})}{k}}$$

$\Rightarrow$ characterizes separable quantum channels

Proof Ideas

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- Classical de Finetti + informationally complete measurements — **relative to quantum side information**

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Sum-of-Squares Hierarchies

[Lasserre 00, Parrilo 03] via information-theoretic approach based on entropy inequalities [Brandão & Harrow 16]

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Sum-of-Squares Hierarchies

[Lasserre 00, Parrilo 03] via information-theoretic approach based on entropy inequalities [Brandão & Harrow 16]

- Various extensions possible — basic open questions for classical/quantum settings

Application: Noisy Channel Coding

Outer Bound Approximations

- Efficiently computable **semi-definite program** outer bounds

$$
\begin{aligned}
\mathrm{sdp}_k(\mathcal{N}, m) := \max \quad & d_{\bar{A}} d_B \cdot \mathrm{Tr}\left[\left(\mathcal{N}_{\bar{A}\to B_1}\left(\Phi_{\bar{A}\bar{A}}\right) \otimes \Phi_{A\bar{B}_1}\right) W_{A\bar{A}B_1\bar{B}_1}\right] \\
\text{s.t.} \quad & W_{A\bar{A}(B\bar{B})_1^k} \geq 0,\ \mathrm{Tr}\left[W_{A\bar{A}(B\bar{B})_1^k}\right] = 1 \\
& W_{A\bar{A}(B\bar{B})_1^k} = \pi_{(B\bar{B})_1^k}\left(W_{A\bar{A}(B\bar{B})_1^k}\right) \\
& W_{A(B\bar{B})_1^k} = \frac{1_A}{2^m} \otimes W_{(B\bar{B})_1^k} \\
& W_{A\bar{A}(B\bar{B})_1^{k-1}B_k} = W_{A\bar{A}(B\bar{B})_1^{k-1}} \otimes \frac{1_{B_k}}{d_B} \\
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with **approximation guarantee** to quantum channel fidelity

$$|\mathrm{spd}_k(\mathcal{N}, m) - F(\mathcal{N}, m)| \leq \sqrt{\frac{\mathrm{poly}(d_A d_{\bar{A}} d_B d_{\bar{B}})}{k}}$$

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$$

- Previous work: [Matthews 12, Leung & Matthews 15, Tomamichel *et al.* 16, Wang *et al.* 16/17] and [Rozpedek *et al.* 18, Kaur *et al.* 18]

Certifying Optimality of Relaxations

- Compare classical linear program relaxation [Barman & Fawzi 18]

$$p(N, m) \leq \mathrm{lp}(N, m) \leq \frac{1}{1 - e^{-1}} \cdot p(N, m)$$

- No finite approximation guarantee for $F(\mathcal{N}, m) \leq \mathrm{sdp}_k(\mathcal{N}, m)$

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Rank Loop Conditions

If for $k \in \mathbb{N}$ there exists $l \in \mathbb{N}$ such that

$$\mathrm{rank}\left(W_{A\bar{A}(B\bar{B})_1^k}\right) \le \max\left\{\mathrm{rank}\left(W_{A\bar{A}(B\bar{B})_1^l}\right), \mathrm{rank}\left(W_{(B\bar{B})_1^{k-l}}\right)\right\}$$

then we have equality $\mathrm{sdp}_k(\mathcal{N},m) = F(\mathcal{N},m)$

- Proof via [Navascués *et al.* 09]

Numerical Example Relaxations

- Uniform noise corresponds to **qubit depolarizing channel**

$$\mathrm{Dep}_p : \rho \mapsto p \cdot \frac{1_B}{2} + (1-p) \cdot \rho \quad \text{with } p \in [0, 4/3].$$

Question

What is the optimal code for reliably storing $m = 1$ qubit in noisy 5 qubit quantum memory, $p\left(\mathrm{Dep}_p^{\otimes 5}, 1\right) = ?$

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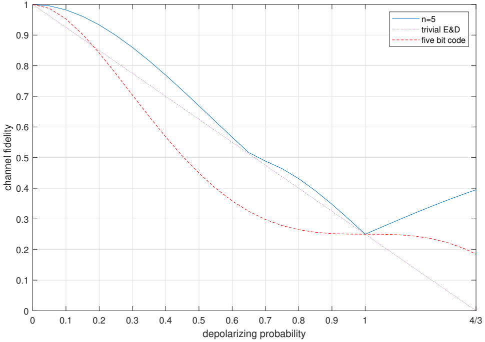
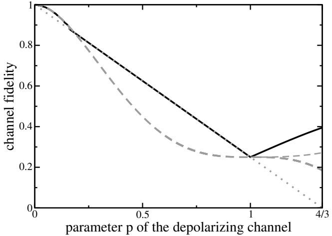
- Analytical [Bennett *et al.* 96] as well as see-saw [Reimpell & Werner 05] lower bounds, our work upper bounds

$$p\left(\mathrm{Dep}_p^{\otimes 5}, 1\right) \leq \mathrm{sdp}_k\left(\mathrm{Dep}_p^{\otimes 5}, 1\right)$$

Numerical Example Relaxations (continued)

- Exploiting symmetries for analytical **dimension reduction** for first level $\mathrm{sdp}_1\left(\mathrm{Dep}_p^{\otimes 5}, 1\right)$ [Wang *et al.* 16/17] gives





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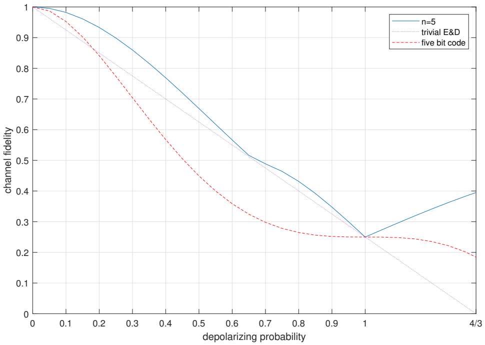

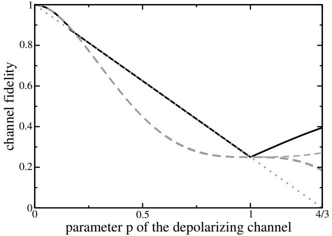

[Reimpell & Werner 05] lower bounds

- $p \in [0, 4/3]$ [Reimpell & Werner 05] optimal, $p \in [0, 0.18]$ room for improved codes

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- Settings with provably good quantum meta-converse?
- Optimal quantum de Finetti theorems: dimension dependence, minimal conditions?

Add-on: De Finetti with Linear Constraints

Quantum De Finetti Theorem with Linear Constraints

Let $\rho_{AB_1^k}$ be a quantum state, $\Lambda_{A\to C_A}, \Gamma_{B\to C_B}$ linear maps, and X_{C_A}, Y_{C_B} operators such that for $\pi \in \mathfrak{S}_k$

$$\pi^{\pi}_{B_1^k}(\rho_{AB_1^k}) = \rho_{AB_1^k} \qquad \text{symmetric with respect to } A$$

$$\Lambda_{A\to C_A}(\rho_{AB_1^k}) = X_{C_A} \otimes \rho_{B_1^k} \qquad \text{linear constraint on } A$$

$$\Gamma_{B_k\to C_B}(\rho_{B_1^n}) = \rho_{B_1^{k-1}} \otimes Y_{C_B} \qquad \text{linear constraint on } B.$$

Then, we have

$$\left\| \rho_{AB} - \sum_{i\in I} p_i \sigma_A^i \otimes \omega_B^i \right\|_1 \leq \sqrt{\frac{d_B^4 (d_B + 1)^2 \log d_A}{k}}$$

with probabilities $\{p_i\}_{i\in I}$ and quantum states σ_A^i, ω_B^i such that $\forall i \in I$

$$\Lambda_{A\to C_A}\left(\sigma_A^i\right) = X_{C_A} \quad \text{and} \quad \Gamma_{B\to C_B}\left(\omega_B^i\right) = Y_{C_B}.$$

Application: Bilinear Optimization

De Finetti with linear constraints gives **outer hierarchy for programs of the bilinear form** (cf. [Huber *et al.* 18])

$$\begin{aligned} \max \quad & \mathrm{Tr}\big[H(D \otimes E)\big] \\ \text{s.t.} \quad & D \in \mathcal{S}_D, E \in \mathcal{S}_E \end{aligned}$$

where H is a matrix and $\mathcal{S}_D$ and $\mathcal{S}_E$ are positive semi-definite representable sets of the form

$$\mathcal{S}_D = \Pi_{A\to D}(\mathcal{S}_A^+ \cap \mathcal{A}_A) \quad \text{and} \quad \mathcal{S}_E = \Pi_{B\to E}(\mathcal{S}_B^+ \cap \mathcal{A}_B)$$

with $\Pi_{A\to D}, \Pi_{B\to E}$ linear maps, $\mathcal{S}_A^+, \mathcal{S}_B^+$ the set of density operators, and $\mathcal{A}_A, \mathcal{A}_B$ affine subspaces of matrices.