Catalan Functions and *k*-Schur Functions

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Joint work with Jennifer Morse, Jonah Blasiak, and Dan Summers

BIRS workshop:
Representation Theory Connections to (q,t)-Combinatorics

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$$H_\mu(\mathbf{x};q,t) = \sum_\lambda K_{\lambda\mu}(q,t)s_\lambda(\mathbf{x}) \qquad \text{for } K_{\lambda\mu}(q,t) \in \mathbb{N}[q,t].$$

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Theorem (Haiman 2001)

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many questions arising in this study remain unanswered.

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Is there a combinatorial interpretation of the coefficients?

Birth of k-Schur functions

20 - 21 years ago ...

Birth of *k*-Schur functions

Photo by snazzyshirtz.com

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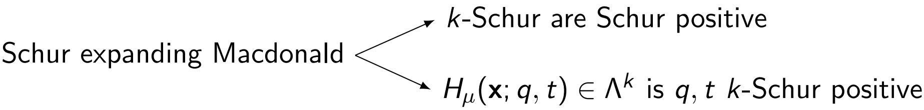

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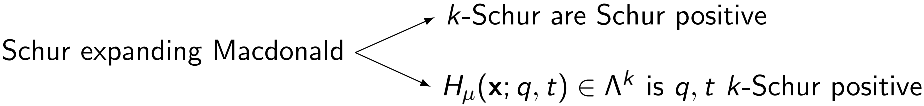

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The intricate construction of these functions lacked in mechanism for proof.

Many conjecturally equivalent candidates have since been proposed, now all informally called *k-Schur functions.*

Strengthened Macdonald positivity conjecture

Example. $k = 2$,

$\Lambda^2 = \mathrm{span}_{\mathbb{Q}(q,t)}\{H_1, H_{11}, H_2, H_{111}, H_{21}, H_{1^4}, H_{211}, H_{22}, \cdots\}$

$$H_{1^4} = t^4\left(s_{\ydiagram{2,2}} + ts_{\ydiagram{3,1}} + t^2 s_{\ydiagram{4}}\right) + (t^2 + t^3)\left(s_{\ydiagram{2,1,1}} + ts_{\ydiagram{3,1}}\right) + \left(s_{\ydiagram{1,1,1,1}} + ts_{\ydiagram{2,1,1}} + t^2 s_{\ydiagram{2,2}}\right)$$

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basis for restricted span Λ^k of Macdonald polynomials

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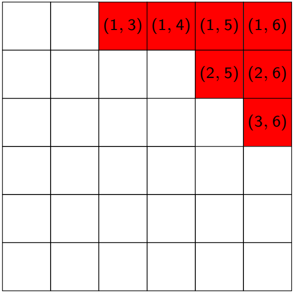

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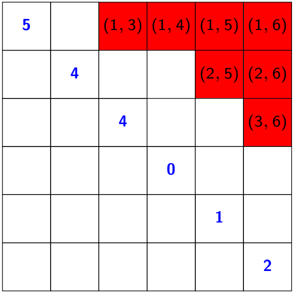

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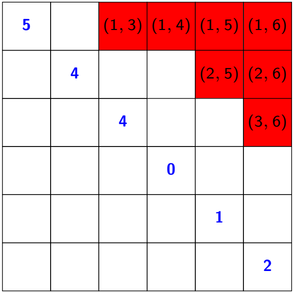

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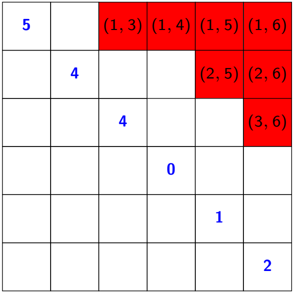

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$$\text{band}(\Psi, \mu) = (6,6,6,2,2,2).$$

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Special cases

Example. Let $\mu = (\mu_1, \ldots, \mu_\ell)$ be a partition.

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4					
	3				
		2			
			2		
				2	
					1

$$H_{432221}^\varnothing(\mathbf{x}; t) = s_{432221}.$$

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4					
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		2			
			2		
				2	
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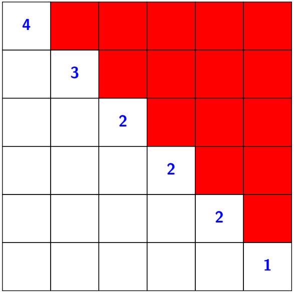

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k-Schur Catalan functions

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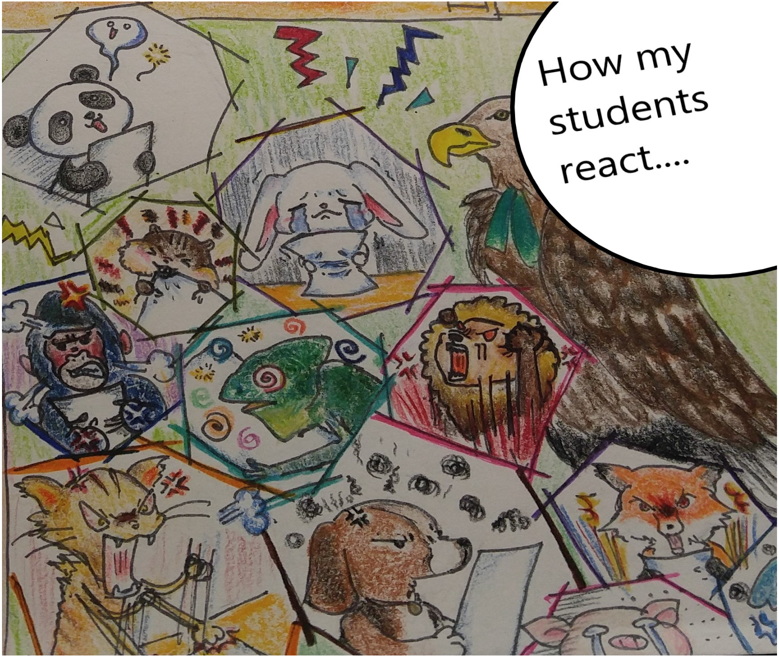
band$(\Delta^k(\mu), \mu)$ is a decreasing sequence

whose first $\ell(\Delta^k(\mu))$ entries are all k


How my students react....

Examples of Catalan functions

Example. $k = 4$, $\mu = 3321$.

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3			
	3		
		2	
			1

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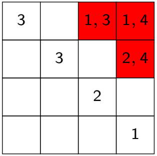

3		1, 3	1, 4
	3		2, 4
		2	
			1

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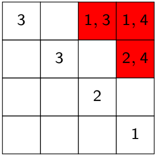

3		1, 3	1, 4
	3		2, 4
		2	
			1

$$\text{band}(\Psi, 3321) = 4431.$$

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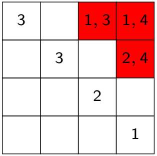

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		2	
			1

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$$\mathfrak{s}_\mu^{(k)}(\mathbf{x}; t) = \prod_{(i,j)\in\Delta^k(\mu)} (1 - tR_{ij})^{-1} s_\mu(\mathbf{x})$$

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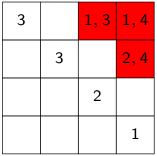

3		1, 3	1, 4
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$$\begin{aligned} \mathfrak{s}_{\mu}^{(k)}(\mathbf{x}; t) &= \prod_{(i,j)\in\Delta^k(\mu)} (1 - tR_{ij})^{-1} s_{\mu}(\mathbf{x}) \\ &= (1 - tR_{13})^{-1}(1 - tR_{24})^{-1}(1 - tR_{14})^{-1} s_{3321}(\mathbf{x}) \end{aligned}$$

Examples of Catalan functions

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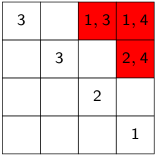

3		1, 3	1, 4
	3		2, 4
		2	
			1

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\mathfrak{s}_{\mu}^{(k)}(\mathbf{x}; t) &= \prod_{(i,j)\in\Delta^k(\mu)} (1 - tR_{ij})^{-1} s_{\mu}(\mathbf{x}) \\
&= (1 - tR_{13})^{-1}(1 - tR_{24})^{-1}(1 - tR_{14})^{-1} s_{3321}(\mathbf{x}) \\
&= s_{3321} + t(s_{3420} + s_{4311} + s_{4320}) + t^2(s_{4410} + s_{5301} + s_{5310}) \\
&\quad + t^3(s_{63-11} + s_{5400} + s_{6300}) + t^4(s_{64-10} + s_{73-10})
\end{aligned}$$

Examples of Catalan functions

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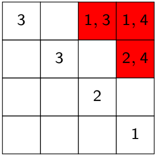

3		1, 3	1, 4
	3		2, 4
		2	
			1

band(Ψ, 3321) = 4431.

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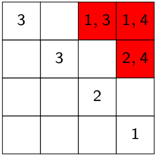

3		1, 3	1, 4
	3		2, 4
		2	
			1

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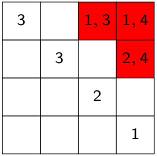

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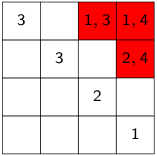

3		1, 3	1, 4
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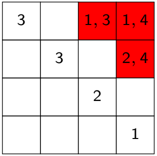

3		1, 3	1, 4
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		2	
			1

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&= s_{3321} + t(s_{4320} + s_{4311}) + t^2(s_{4410} + s_{5310}) + t^3 s_{5400}.
\end{aligned}$$

Examples of Catalan functions

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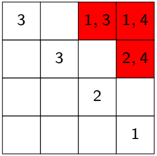

3		1, 3	1, 4
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		2	
			1

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&= s_{3321} + t(0 + s_{4311} + s_{4320}) + t^2(s_{4410} + 0 + s_{5310}) \\
&\quad + t^3(-s_{6300} + s_{5400} + s_{6300}) + t^4(0 + 0) \\
&= 1 \cdot s_{3321} + t(s_{4320} + s_{4311}) + t^2(s_{4410} + s_{5310}) + t^3 s_{5400}.
\end{aligned}$$


THIS is
AMAZING

Is there any combinatorial objects related?

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YES! :)

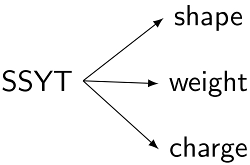
From SSYT to SMT

From SSYT to SMT

From SSYT to SMT

SSYT

From SSYT to SMT



SSYT

shape

weight

charge

From SSYT to SMT

1	1	1	1	3
2	2	3	4	
4				

shape $= 541$

weight

charge

From SSYT to SMT

1	1	1	1	3
2	2	3	4	
4				

shape $= 541$

weight $= 4222$

charge

From SSYT to SMT

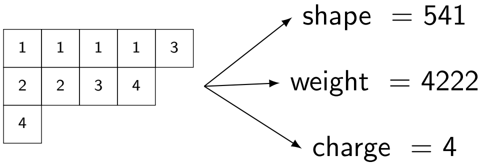
1	1	1	1	3
2	2	3	4	
4				

shape $= 541$

weight $= 4222$

charge $= 4$

From SSYT to SMT



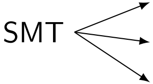
1	1	1	1	3
2	2	3	4	
4				

shape $= 541$

weight $= 4222$

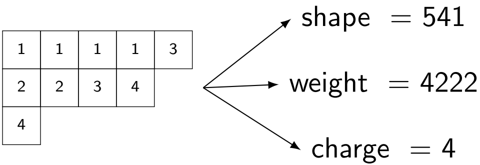
charge $= 4$

Fix a positive integer k (for construction),



SMT

From SSYT to SMT



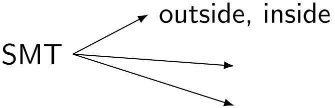
1	1	1	1	3
2	2	3	4	
4				

shape $= 541$

weight $= 4222$

charge $= 4$

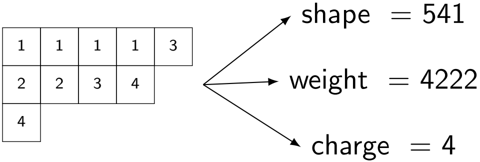
Fix a positive integer k (for construction),



SMT

outside, inside

From SSYT to SMT



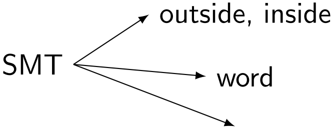
1	1	1	1	3
2	2	3	4	
4				

shape $= 541$

weight $= 4222$

charge $= 4$

Fix a positive integer k (for construction),

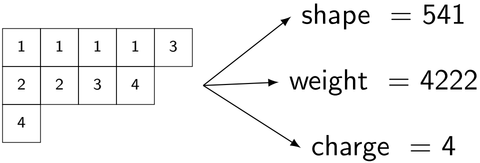


SMT

outside, inside

word

From SSYT to SMT



1	1	1	1	3
2	2	3	4	
4				

shape $= 541$

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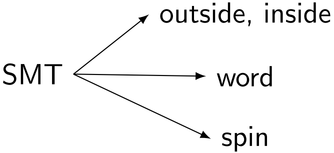
charge $= 4$

Fix a positive integer k (for construction),



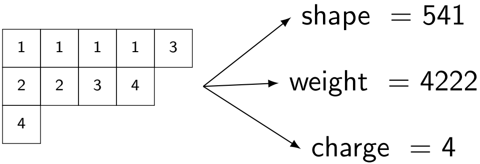
SMT

outside, inside

word

spin

From SSYT to SMT



1	1	1	1	3
2	2	3	4	
4				

shape $= 541$

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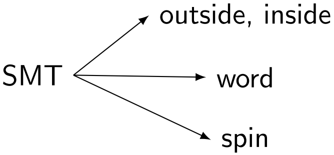
charge $= 4$

Fix a positive integer $k = 4$,



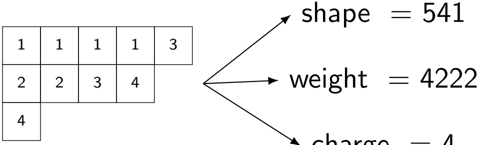
SMT

outside, inside

word

spin

From SSYT to SMT



1	1	1	1	3
2	2	3	4	
4				

shape $= 541$

weight $= 4222$

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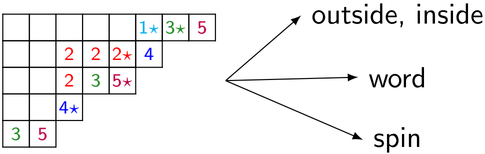
Fix a positive integer $k = 4$,



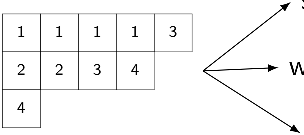
					$1\star$	$3\star$	5
		2	2	$2\star$	4		
		2	3	$5\star$			
		$4\star$					
3	5						

outside, inside

word

spin

From SSYT to SMT



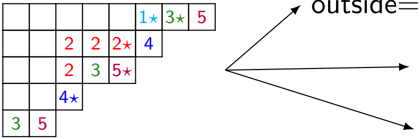
1	1	1	1	3
2	2	3	4	
4				

shape = 541

weight = 4222

charge = 4

Fix a positive integer $k = 4$,



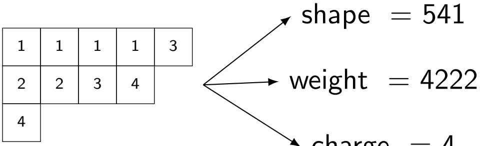
					1⋆	3⋆	5
		2	2	2⋆	4		
		2	3	5⋆			
		4⋆					
3	5						

outside= 33332, inside

word

spin

From SSYT to SMT

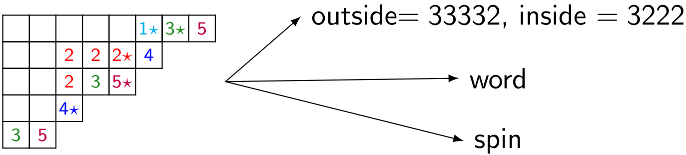


shape $= 541$

weight $= 4222$

charge $= 4$

Fix a positive integer $k = 4$,

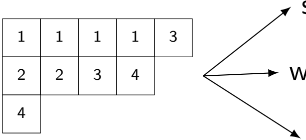


outside$= 33332$, inside $= 3222$

word

spin

From SSYT to SMT



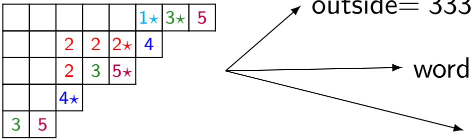
1	1	1	1	3
2	2	3	4	
4				

shape $= 541$

weight $= 4222$

charge $= 4$

Fix a positive integer $k \; = 4$,



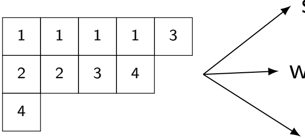
					$1\star$	$3\star$	5
		2	2	$2\star$	4		
		2	3	$5\star$			
		$4\star$					
3	5						

outside$= 33332$, inside $= 3222$

word $= 34121$

spin

From SSYT to SMT



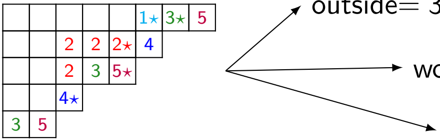
1	1	1	1	3
2	2	3	4	
4				

shape $= 541$

weight $= 4222$

charge $= 4$

Fix a positive integer $k = 4$,



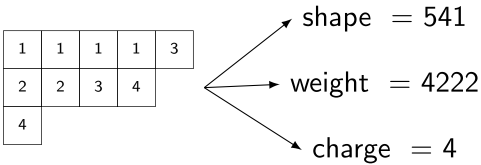
					$1\star$	$3\star$	5
		2	2	$2\star$	4		
		2	3	$5\star$			
		$4\star$					
3	5						

outside$= 33332$, inside $= 3222$

word $= 34121$

spin $= 4$

From SSYT to SMT



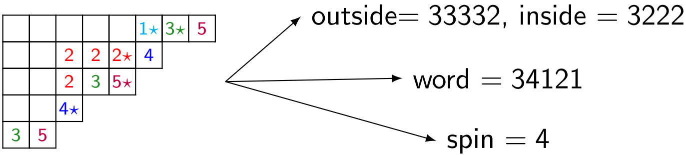
1	1	1	1	3
2	2	3	4	
4				

shape $= 541$

weight $= 4222$

charge $= 4$

Fix a positive integer $k = 4$,



					$1\star$	$3\star$	5
		2	2	$2\star$	4		
		2	3	$5\star$			
		$4\star$					
3	5						

outside$= 33332$, inside $= 3222$

word $= 34121$

spin $= 4$

We write $\mathrm{SMT}^k(w\,;\mu) =$ set of strong tableaux T marked by w with outside$(T) = \mu$.

Strong Pieri Operators

Def. Fix a positive integer k. The *strong Pieri operators* $u_1, u_2, \cdots \in \mathrm{End}_{\mathbb{Z}[t]}(\Lambda^k)$ are defined by their action on the basis $\{\mathfrak{s}_\mu^{(k)}\}_{\mu \in \mathsf{Par}^k}$ as follows:

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Example.

Strong Pieri Operators

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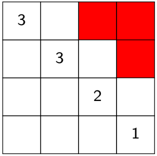

$\cdot\ u_2$

$$\mathfrak{s}_{3321}^{(4)} \cdot u_2$$

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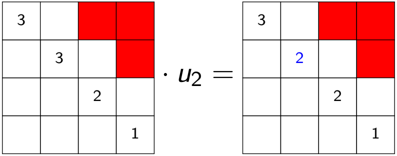

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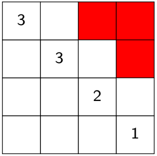
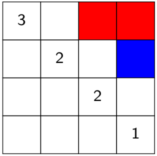


$\cdot\, u_2 =$



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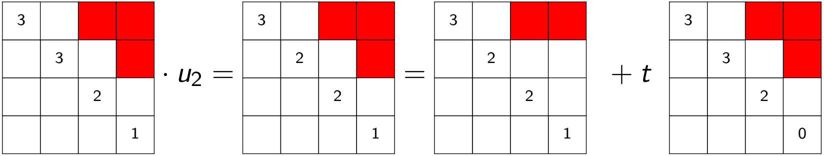

$$\mathfrak{s}_{3321}^{(4)} \cdot u_2 = \mathfrak{s}_{3221}^{(4)} + t\mathfrak{s}_{3320}^{(4)}.$$

Properties of k-Schur functions

Theorem (Blasiak-Morse-P.-Summers)

The k-Schur functions $\{\mathfrak{s}_\mu^{(k)} \mid \mu$ *is k-bounded of length* $\leq \ell\}$ *satisfy*

(vertical dual Pieri rule) $$e_d^\perp \mathfrak{s}_\mu^{(k)} = \mathfrak{s}_\mu^{(k)} \cdot \left(\sum_{i_1 > \cdots > i_d} u_{i_1} \cdots u_{i_d} \right),$$

(shift invariance) $$\mathfrak{s}_\mu^{(k)} = e_\ell^\perp \mathfrak{s}_{\mu+1^\ell}^{(k+1)},$$

(Schur function stability) *if* $k \geq |\mu|$, *then* $\mathfrak{s}_\mu^{(k)} = s_\mu$.

- $e_d^\perp \in \mathrm{End}(\Lambda)$ is defined by $\langle e_d^\perp(g), h\rangle = \langle g, e_d h\rangle$ for all $g, h \in \Lambda$.
- u_i = operator for removing a strong cover marked in row i.

k-Schur branching rule

Theorem (Blasiak-Morse-P.-Summers)

For μ a k-bounded partition of length $\leq \ell$, the expansion of the k-Schur function $s_\mu^{(k)}$ into $k+1$-Schur functions is given by

$$\mathfrak{s}_\mu^{(k)} = \mathfrak{s}_{\mu+1^\ell}^{(k+1)} u_\ell \cdots u_1 = \sum_{T \in \mathrm{SMT}^{k+1}(\ell \cdots 2\,1\,;\,\mu+1^\ell)} t^{\mathrm{spin}(T)} \mathfrak{s}_{\mathrm{inside}(T)}^{(k+1)} .$$

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Proof.

The shift invariance property followed by the vertical dual Pieri rule yields

$$\mathfrak{s}_\mu^{(k)} = e_\ell^\perp \, \mathfrak{s}_{\mu+1^\ell}^{(k+1)} = \mathfrak{s}_{\mu+1^\ell}^{(k+1)} u_\ell \cdots u_1 .$$

□

k-Schur branching rule

$$\mathfrak{s}^{(3)}_{22221} = t^3\mathfrak{s}^{(4)}_{3321} + t^2\mathfrak{s}^{(4)}_{3222} + t^2\mathfrak{s}^{(4)}_{33111} + \mathfrak{s}^{(4)}_{22221}$$

					$1\star$	3	5
				$2\star$	4		
		1	$3\star$	5			
	2	$4\star$					
3	$5\star$						

					$1\star$	3	5
		2	2	$2\star$	4		
		2	$3\star$	5			
		$4\star$					
3	$5\star$						

				$1\star$	3	3	5
				$2\star$	4		
	1	3	$3\star$	5			
	2	$4\star$					
	$5\star$						

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			2	$2\star$	4		
		3	$3\star$	5			
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$\mathrm{SMT}^4(54321; 33332)$

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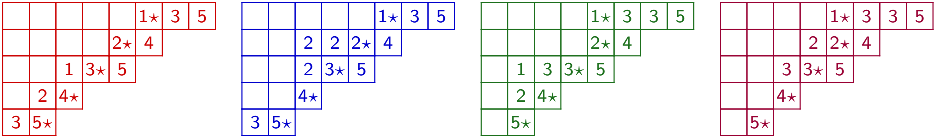
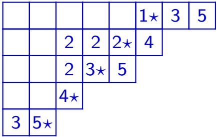
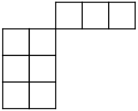
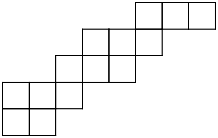


$\mathrm{SMT}^4(54321; 33332)$

$T =$







$\mathrm{spin}(T) = 0 + 1 + 1 + 0 + 0 = 2$ $\quad$ $\mathrm{inside}(T) = 3222$ $\quad$ $\mathrm{outside}(T) = 33332$

k-Schur into Schur

Theorem (Blasiak-Morse-P.-Summers)

Let μ be a k-bounded partition of length $\leq \ell$ and set $m = \max(|\mu| - k, 0)$. The Schur expansion the k-Schur function $\mathfrak{s}_\mu^{(k)}$ is given by

$$\mathfrak{s}_\mu^{(k)} = \sum_{T \in \mathrm{SMT}^{k+m}(\,(\ell \cdots 1)^m\,;\,\mu + m^\ell\,)} t^{\mathrm{spin}(T)} s_{\mathrm{inside}(T)}.$$

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Proof.

Applying the shift invariance property m times followed by the vertical dual Pieri rule, we obtain

$$\mathfrak{s}_\mu^{(k)} = (e_\ell^\perp)^m \, \mathfrak{s}_{\mu+m^\ell}^{(k+m)} = \mathfrak{s}_{\mu+m^\ell}^{(k+m)} (u_\ell \cdots u_1)^m = \sum_{T \in \mathrm{SMT}^{k+m}((\ell\cdots 1)^m;\mu+m^\ell)} t^{\mathrm{spin}(T)} s_{\mathrm{inside}(T)}.$$

The Schur function stability property ensures this is the Schur function decomposition. $\square$

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Example. $k = 1, \mu = 111,$

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$$\mathrm{SMT}^{k+m}((\ell\cdots 1)^m; \mu + m^\ell) = \mathrm{SMT}^3((321)^2; 111 + 2^3)$$

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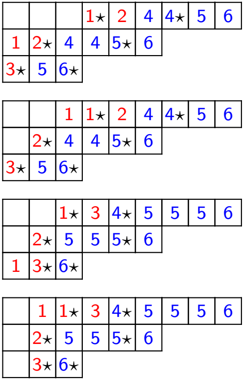
Schur expansion of $s_{111}^{(1)} = H_{111}$



			1⋆	2	4	4⋆	5	6
1	2⋆	4	4	5⋆	6			
3⋆	5	6⋆						

$t^3\, s_3$

		1	1⋆	2	4	4⋆	5	6
	2⋆	4	4	5⋆	6			
3⋆	5	6⋆						

$t^2\, s_{21}$

		1⋆	3	4⋆	5	5	5	6
	2⋆	5	5	5⋆	6			
1	3⋆	6⋆						

$t\, s_{21}$

	1	1⋆	3	4⋆	5	5	5	6
	2⋆	5	5	5⋆	6			
	3⋆	6⋆						

s_{111}

$$\mathfrak{s}_{111}^{(1)} = t^3 s_3 + t^2 s_{21} + t s_{21} + s_{111}$$

The Schur expansion of the 1-Schur function $\mathfrak{s}_{111}^{(1)}$ is obtained by summing $t^{\mathrm{spin}(T)} s_{\mathrm{inside}(T)}$ over the set $\mathrm{SMT}^3(321321; 333)$ of strong tableaux T above.

Unifying the definitions of k-Schur functions

- $s_{\mu}^{(k)}(\mathbf{x}; t)$ defined as a sum of monomials over strong tableaux. Equivalent to the symmetric functions satisfying the dual Pieri rule.

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Unifying the definitions of k-Schur functions

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Combining our results with those of Lam and Lam-Lapointe-Morse-Shimozono:

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Combining our results with those of Lam and Lam-Lapointe-Morse-Shimozono:

Theorem

The k-Schur functions defined from Jing vertex operators, k-Schur Catalan functions, and strong tableau k-Schur functions coincide:

$$\tilde{A}_\mu^{(k)}(\mathbf{x};t) = \mathfrak{s}_\mu^{(k)}(\mathbf{x};t) = s_\mu^{(k)}(\mathbf{x};t) \quad \text{for all } k\text{-bounded } \mu.$$

Moreover, their $t = 1$ *specializations* $\{s_\mu^{(k)}(\mathbf{x};1)\}$ *match a definition using weak tableaux, and represent Schubert classes in the homology of the affine Grassmannian* Gr_G *of* $G = SL_{k+1}$.

k-Schur positivity of Catalan Functions

Proposition

If (Ψ, μ) *is an indexed root ideal with* $\mathrm{band}(\Psi, \mu)_i \leq k$ *for all* i*, then* $H(\Psi; \mu) \in \Lambda^k$*.*

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Are they k-Schur positive?

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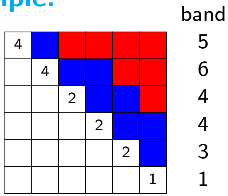

This Catalan function is 6-Schur positive.

k-Schur positivity of Catalan Functions

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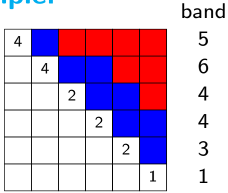

						band
4	🟦	🟥	🟥	🟥	🟥	5
	4	🟦	🟦	🟥	🟥	6
		2	🟦	🟦	🟥	4
			2	🟦	🟦	4
				2	🟦	3
					1	1

This Catalan function is 6-Schur positive.

Conjecture (Chen-Haiman)

The Catalan function H_μ^Ψ *is Schur positive for any root ideal* Ψ *and partition* μ*.*

Special cases

Let $\mu \in \mathsf{Par}^k$.

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- product of a schur and a k-schur (and also its generalization) when the indexing partition concatenate to a partition;

.

Special cases

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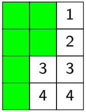

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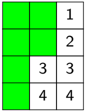

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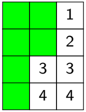
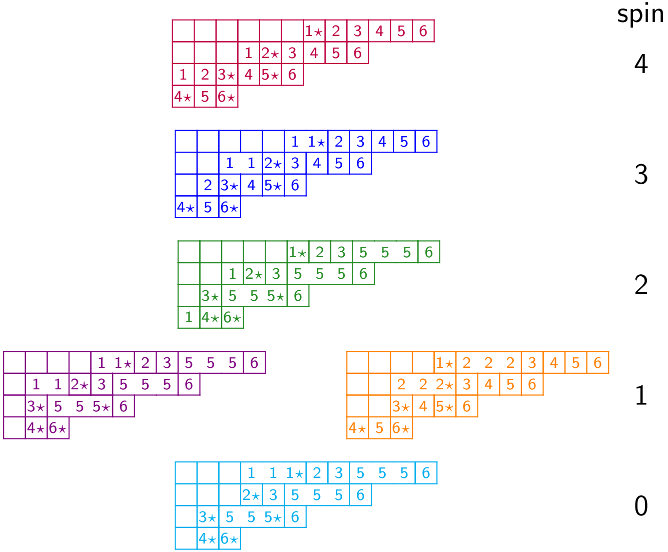


$$Z_{(3333)/(2211)} = \begin{array}{|c|c|c|} \hline \blacksquare & \blacksquare & 1 \\ \hline \blacksquare & \blacksquare & 2 \\ \hline \blacksquare & 3 & 3 \\ \hline \blacksquare & 4 & 4 \\ \hline \end{array} \qquad \text{and} \qquad \mathrm{colword}(Z_{(3333)/(2211)}) = 434321\,.$$

$$H_{2211} = \mathfrak{s}^{(3)}_{3333} \cdot u_4 u_3 u_4 u_3 u_2 u_1 = \sum_{\mathrm{SMT}^3(434321\,;\,3333)} t^{\mathrm{spin}(T)} \mathfrak{s}^{(3)}_{\mathrm{inside}(T)}$$

The 3-Schur expansion of H_{2211}



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spin

						1⋆	2	3	4	5	6
			1	2⋆	3	4	5	6			
1	2	3⋆	4	5⋆	6						
4⋆	5	6⋆									

4

inside = 321

spin = 1+1+0+0+1+0

		■	■	■	1	1⋆	2	3	4	5	6
■	■	1	1	2⋆	3	4	5	6			
■	2	3⋆	4	5⋆	6						
4⋆	5	6⋆									

3

					1⋆	2	3	5	5	5	6
		1	2⋆	3	5	5	5	6			
	3⋆	5	5	5⋆	6						
1	4⋆	6⋆									

2

				1	1⋆	2	3	5	5	5	6
	1	1	2⋆	3	5	5	5	6			
	3⋆	5	5	5⋆	6						
	4⋆	6⋆									

				1⋆	2	2	2	3	4	5	6
		2	2	2⋆	3	4	5	6			
		3⋆	4	5⋆	6						
4⋆	5	6⋆									

1

			1	1	1⋆	2	3	5	5	5	6
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0

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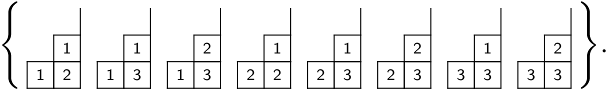

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$$\mathbf{B}_\mu \, \mathfrak{s}_\nu^{(k)} = \mathfrak{s}_{R\nu}^{(k)} \cdot \left(u_{121} + u_{131} + u_{132} + u_{221} + u_{231} + u_{232} + u_{331} + u_{332} \right).$$

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$$\mathbf{B}_{432} \, \mathfrak{s}_{22}^{(6)} = \mathfrak{s}_{44422}^{(6)} \cdot \big(u_{121} + u_{131} + u_{132} + u_{221} + u_{231} + u_{232} + u_{331} + u_{332} \big).$$

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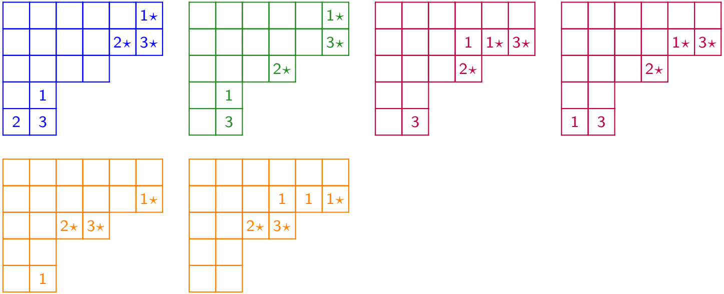
Example. 6-Schur expansion of a t-analog of $s_{432}\, s_{22}$.

$$\mathbf{B}_{432}\, \mathfrak{s}^{(6)}_{22} = \mathfrak{s}^{(6)}_{44422} \cdot \left(u_{121} + u_{131} + u_{132} + u_{221} + u_{231} + u_{232} + u_{331} + u_{332} \right).$$



$$\mathbf{B}_{432}\, \mathfrak{s}^{(6)}_{22} = t^3 \mathfrak{s}^{(6)}_{4441} + t^2 \mathfrak{s}^{(6)}_{44311} + t^2 \mathfrak{s}^{(6)}_{4432} + t^1 \mathfrak{s}^{(6)}_{43321} + t^1 \mathfrak{s}^{(6)}_{44221} + \mathfrak{s}^{(6)}_{43222}.$$

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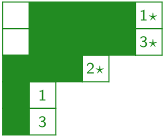

inside = 44311
spin = 1 + 0 + 1 = 2

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peace
love
Thank You for listening

k-skew diagrams

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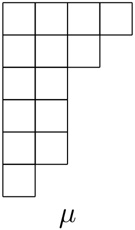

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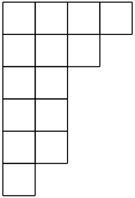

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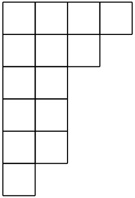

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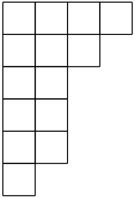

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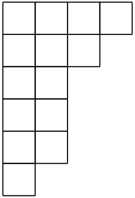

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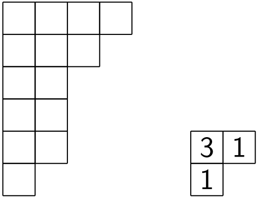

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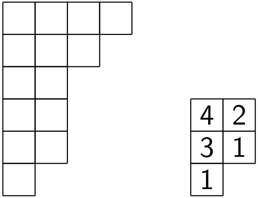

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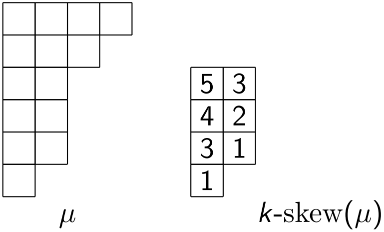

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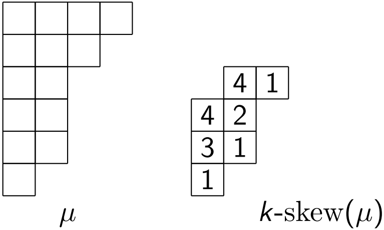

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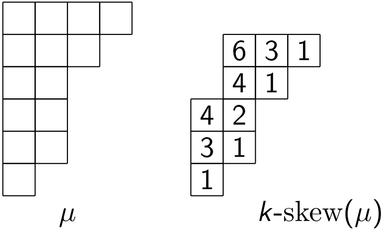

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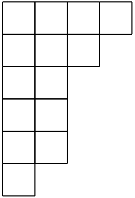
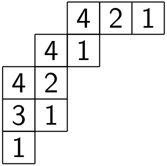


μ



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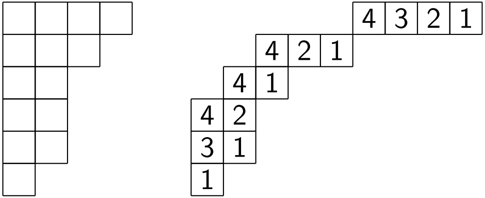

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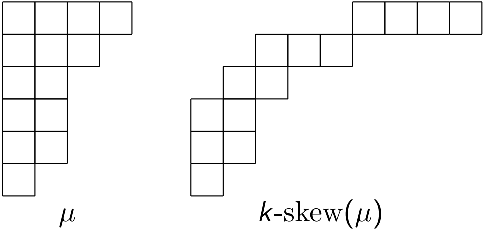

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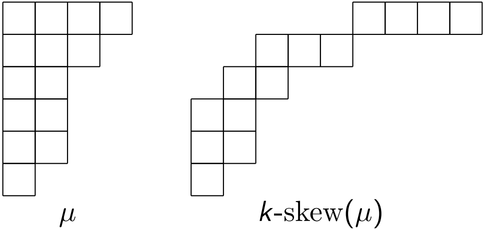
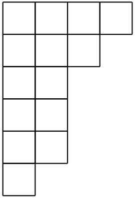


$k+1$-cores

Def. $\mathfrak{p}(\mu)$ denotes the outer shape of k-$\mathrm{skew}(\mu)$.

Example. $k=4$:



μ

14	12	9	7	6	4	3	2	1
9	7	4	2	1				
6	4	1						
4	2							
3	1							
1								

$\mathfrak{p}(\mu)$

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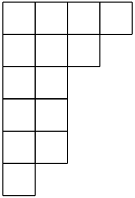

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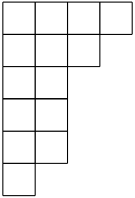

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1								

$\mathfrak{p}(\mu)$

Proposition. The map $\mu \mapsto \mathfrak{p}(\mu)$ defines a bijection from k-bounded partitions to $k+1$-cores.

Strong covers

Def. An inclusion $\tau \subset \kappa$ of $k+1$-cores is a *strong cover*, denoted $\tau \Rightarrow \kappa$, if $|\mathfrak{p}^{-1}(\tau)| + 1 = |\mathfrak{p}^{-1}(\kappa)|$.

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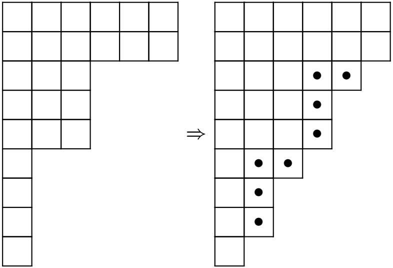
Strong cover with $k = 4$:



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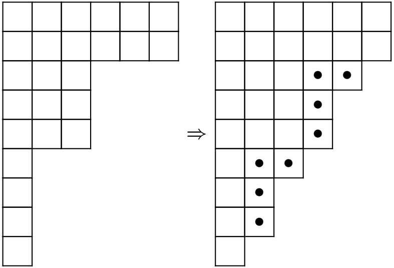
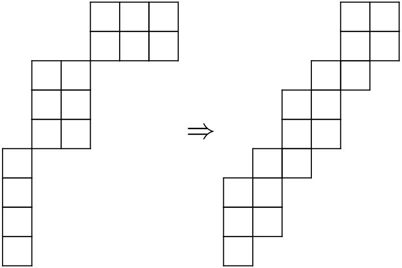


corresponding k-skew diagrams:



$\mathfrak{p}^{-1}(\tau) = 332221111$ $\quad$ $\mathfrak{p}^{-1}(\kappa) = 222222221$

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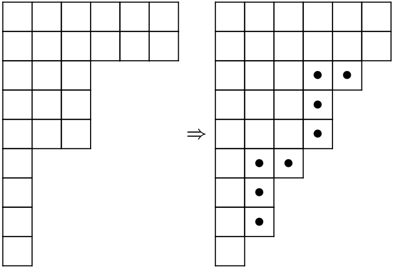

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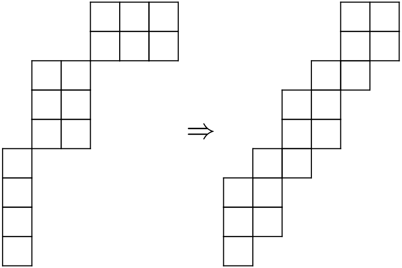


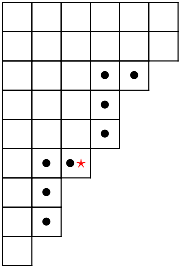
$|\mathfrak{p}^{-1}(\tau)| = 16$

$|\mathfrak{p}^{-1}(\kappa)| = 17$

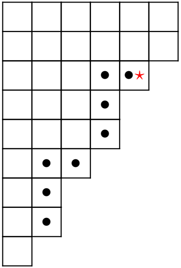
Strong marked covers

Def. A *strong marked cover* $\tau \xRightarrow{r} \kappa$ is a strong cover $\tau \Rightarrow \kappa$ together with a positive integer r which is allowed to be the smallest row index of any connected component of the skew shape κ/τ.

Example. The two possible markings of the previous strong cover:



$\tau \xRightarrow{6} \kappa$



$\tau \xRightarrow{3} \kappa$

Spin

Def.

$$spin(\tau \xRightarrow{r} \kappa) = c \cdot (h-1) + N, \quad \text{where}$$

- c = number of connected components of κ/τ,
- h = height (number of rows) of each component,
- N = number of components below the marked one.

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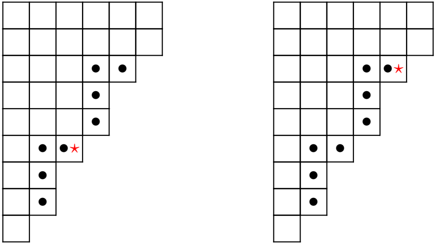

$\tau \xrightarrow{6} \kappa$ $\tau \xrightarrow{3} \kappa$

spin $= 4$ spin $= 5$

spin $= c \cdot (h-1) + N = 2 \cdot (3-1) + 0 = 4$ spin $= 2 \cdot (3-1) + 1 = 5$

Strong marked tableaux

Def. For a word $w = w_1 \cdots w_m \in \mathbb{Z}_{\geq 1}^m$, a *strong tableau marked by w* is a sequence of strong marked covers of the form

$$\kappa^{(0)} \xRightarrow{w_m} \kappa^{(1)} \xRightarrow{w_{m-1}} \cdots \xRightarrow{w_1} \kappa^{(m)}.$$

- inside$(T) := \mathfrak{p}^{-1}(\kappa^{(0)})$
- outside$(T) := \mathfrak{p}^{-1}(\kappa^{(m)})$

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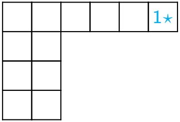
Example. For $k = 4$, a strong marked tableau marked by 34121:



$$\kappa^{(0)} \xRightarrow{1} \kappa^{(1)}, \text{ spin} = 1(1-1) + 0 = 0$$

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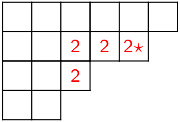

$$\kappa^{(1)} \xRightarrow{2} \kappa^{(2)}, \text{ spin} = 1(2-1) + 0 = 1$$

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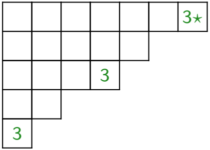

$$\kappa^{(2)} \xRightarrow{1} \kappa^{(3)}, \text{ spin} = 3(1-1) + 2 = 2$$

Strong marked tableaux

Def. For a word $w = w_1 \cdots w_m \in \mathbb{Z}_{\geq 1}^m$, a *strong tableau marked by w* is a sequence of strong marked covers of the form

$$\varkappa^{(0)} \xRightarrow{w_m} \varkappa^{(1)} \xRightarrow{w_{m-1}} \cdots \xRightarrow{w_1} \varkappa^{(m)}.$$

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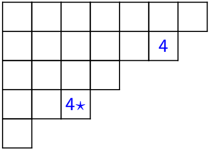

$$\varkappa^{(3)} \xRightarrow{4} \varkappa^{(4)}, \text{ spin} = 2(1-1)+0 = 0$$

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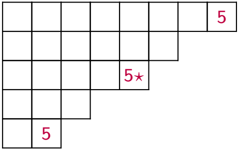

$$\kappa^{(4)} \xRightarrow{3} \kappa^{(5)}, \text{ spin} = 3(1-1)+1 = 1$$

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T =

					$1\star$	$3\star$	5
		2	2	$2\star$	4		
		2	3	$5\star$			
		$4\star$					
3	5						

, $\quad$ spin$(T) = 1 + 0 + 2 + 1 + 0 = 4$

Spin k-Schur functions

- We work in the ring of symmetric functions in infinitely many variables $\mathbf{x} = (x_1, x_2, \dots)$.

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$$s_\mu^{(k)}(\mathbf{x};t) = \sum_{1 \le i_1 \le \cdots \le i_d} \; \sum_{\substack{w \in \mathbb{Z}^d_{\ge 1} \\ i_j = i_{j+1} \implies w_j \le w_{j+1}}} \; \sum_{T \in \mathrm{SMT}^k(w;\mu)} t^{\mathrm{spin}(T)} x_{i_1} \cdots x_{i_d} \,.$$

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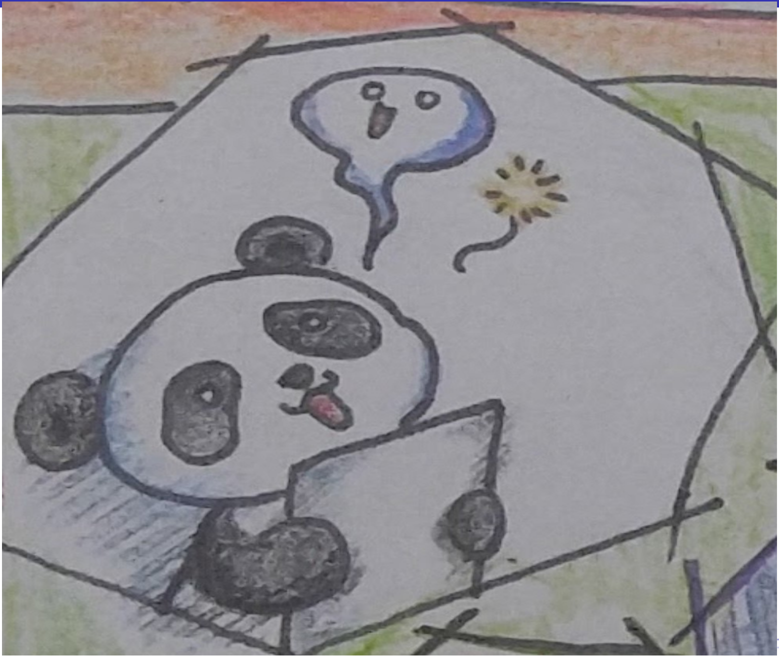
Their $t = 1$ specializations

- agree with another combinatorial definition using weak tableaux (Lam-Lapointe-Morse-Shimozono 2010),
- are Schubert classes in the homology of the affine Grassmannian $\mathrm{Gr}_{SL_{k+1}}$ of SL_{k+1} (Lam 2008).

Spin k-Schur functions



Spin k-Schur functions

Example. $k = 3$, $\mu = 311$:

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There are 10 strong marked standard tableaux T whose 4-core is 411 with outside(T) $= 311$:

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$$\begin{array}{|c|c|c|c|}\hline 1\star & 2\star & 4 & 4\star \\ \hline 3\star \\ \cline{1-1} 5\star \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|c|}\hline 1\star & 2\star & 4 & 5\star \\ \hline 3\star \\ \cline{1-1} 4\star \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|c|}\hline 1\star & 3\star & 4 & 5\star \\ \hline 2\star \\ \cline{1-1} 4\star \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|c|}\hline 1\star & 3\star & 4 & 4\star \\ \hline 2\star \\ \cline{1-1} 5\star \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|c|}\hline 1\star & 2\star & 3\star & 4 \\ \hline 4\star \\ \cline{1-1} 5\star \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|c|}\hline 1\star & 4 & 4\star & 5\star \\ \hline 2\star \\ \cline{1-1} 3\star \\ \cline{1-1}\end{array} \qquad \text{spin} = 0$$

$w = 31211$ $w = 13211$ $w = 13121$ $w = 31121$ $w = 32111$ $w = 11321$

$wt = 221$ $wt = 212$ $wt = 122$ $wt = 131$ $wt = 311$ $wt = 113$

$$\begin{array}{|c|c|c|c|}\hline 1\star & 2\star & 3\star & 4\star \\ \hline 4 \\ \cline{1-1} 5\star \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|c|}\hline 1\star & 2\star & 3\star & 5\star \\ \hline 4\star \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|c|}\hline 1\star & 2\star & 4\star & 5\star \\ \hline 3\star \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|c|}\hline 1\star & 3\star & 4\star & 5\star \\ \hline 2\star \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} \qquad \text{spin} = 1$$

$w = 31111$ $w = 12111$ $w = 11211$ $w = 11121$

$wt = 41$ $wt = 32$ $wt = 23$ $wt = 14$

Spin k-Schur functions

Example. $k = 3$, $\mu = 311$:
There are 10 strong marked standard tableaux T whose 4-core is 411 with outside(T) $= 311$:

$$\begin{array}{|c|c|c|c|}\hline 1\star & 2\star & 4 & 4\star \\ \hline \multicolumn{1}{|c|}{3\star} & \multicolumn{3}{c}{} \\ \cline{1-1} \multicolumn{1}{|c|}{5\star} & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|c|}\hline 1\star & 2\star & 4 & 5\star \\ \hline \multicolumn{1}{|c|}{3\star} & \multicolumn{3}{c}{} \\ \cline{1-1} \multicolumn{1}{|c|}{4\star} & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|c|}\hline 1\star & 3\star & 4 & 5\star \\ \hline \multicolumn{1}{|c|}{2\star} & \multicolumn{3}{c}{} \\ \cline{1-1} \multicolumn{1}{|c|}{4\star} & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|c|}\hline 1\star & 3\star & 4 & 4\star \\ \hline \multicolumn{1}{|c|}{2\star} & \multicolumn{3}{c}{} \\ \cline{1-1} \multicolumn{1}{|c|}{5\star} & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|c|}\hline 1\star & 2\star & 3\star & 4 \\ \hline \multicolumn{1}{|c|}{4\star} & \multicolumn{3}{c}{} \\ \cline{1-1} \multicolumn{1}{|c|}{5\star} & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|c|}\hline 1\star & 4 & 4\star & 5\star \\ \hline \multicolumn{1}{|c|}{2\star} & \multicolumn{3}{c}{} \\ \cline{1-1} \multicolumn{1}{|c|}{3\star} & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array} \quad \text{spin} = 0$$

$w = 31211$ $w = 13211$ $w = 13121$ $w = 31121$ $w = 32111$ $w = 11321$

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$$\begin{array}{|c|c|c|c|}\hline 1\star & 2\star & 3\star & 4\star \\ \hline \multicolumn{1}{|c|}{4} & \multicolumn{3}{c}{} \\ \cline{1-1} \multicolumn{1}{|c|}{5\star} & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|c|}\hline 1\star & 2\star & 3\star & 5\star \\ \hline \multicolumn{1}{|c|}{4\star} & \multicolumn{3}{c}{} \\ \cline{1-1} \multicolumn{1}{|c|}{4} & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|c|}\hline 1\star & 2\star & 4\star & 5\star \\ \hline \multicolumn{1}{|c|}{3\star} & \multicolumn{3}{c}{} \\ \cline{1-1} \multicolumn{1}{|c|}{4} & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|c|}\hline 1\star & 3\star & 4\star & 5\star \\ \hline \multicolumn{1}{|c|}{2\star} & \multicolumn{3}{c}{} \\ \cline{1-1} \multicolumn{1}{|c|}{4} & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array} \quad \text{spin} = 1$$

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$$s^{(3)}_{311} = tm_{41} + tm_{32} + (1+2t)m_{311} + (1+2t)m_{221} + (3+3t)m_{2111} + (6+4t)m_{11111}.$$