Spanning Configurations

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UCSD

Outline

1. Flags
2. Spanning Lines
3. Spanning Subspaces

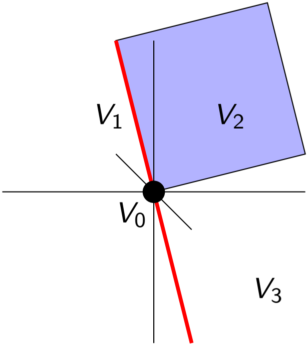
Flags

Def: A *flag* in $\mathbb{C}^n$ is a nested chain of subspaces

$$V_\bullet : (0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n)$$

with $\dim(V_i) = i$.

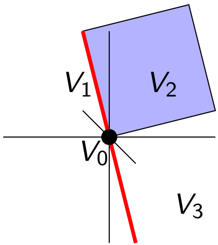
$$\mathcal{F}\ell(n) = \{\text{all flags in } \mathbb{C}^n\} = GL_n/B.$$



$\in \mathcal{F}\ell(3)$

Structure of $\mathcal{F}\ell(n)$

Def: $\mathcal{F}\ell(n) = GL_n/B$.



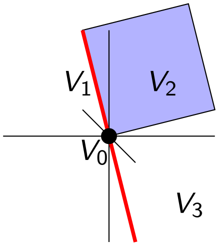
The *(open) Schubert cell* for $w \in S_n$ is $X_w := BwB/B$.

$$\mathcal{F}\ell(n) = \bigsqcup_{w \in S_n} X_w$$

$\Rightarrow$ over $\mathbb{F}_q$, we have $|\mathcal{F}\ell(n)| = [n]!_q$.

Structure of $\mathcal{F}\ell(n)$

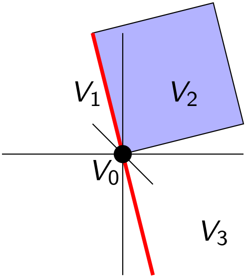
Def: $\mathcal{F}\ell(n) = \{\text{all flags in } \mathbb{C}^n\} = GL_n(\mathbb{C})/B$.



Thm: [Ehresmann] The cells $\{X_w : w \in S_n\}$ induce a CW decomposition of $\mathcal{F}\ell(n)$.
$\Rightarrow$ The *Poincaré series* of $\mathcal{F}\ell(n)$ is $[n]!_{q^2}$.

Cohomology of $\mathcal{F}\ell(n)$

Def: $\mathcal{F}\ell(n) = \{\text{all flags in } \mathbb{C}^n\} = GL_n(\mathbb{C})/B$.



Thm: [Borel] $H^\bullet(\mathcal{F}\ell(n)) = \mathbb{Z}[\mathbf{x}_n]/\langle e_1, e_2, \ldots, e_n\rangle =: R_n$.
Here $x_i \leftrightarrow -c_1(V_i/V_{i-1})$.

Rmk: True as graded rings or graded S_n-modules.

Reminders on R_n

$$R_n = \mathbb{Z}[\mathbf{x}_n]/\langle e_1, e_2, \ldots, e_n\rangle = H^\bullet(\mathcal{F}\ell(n)).$$

Thm: [Chevalley] As *ungraded* S_n-modules:

$$R_n^{\mathbb{Q}} \cong \mathbb{Q}[S_n]$$

Thm: [Lusztig-Stanley] The *graded* structure is given by

$$\mathrm{grFrob}(R_n^{\mathbb{Q}}; q) = \sum_{T \in \mathrm{SYT}(n)} q^{\mathrm{maj}(T)} s_{\mathrm{shape}(T)}$$

Schubert polynomials

The *Schubert polynomials* $\{\mathfrak{S}_w : w \in S_n\}$ are recursively defined by

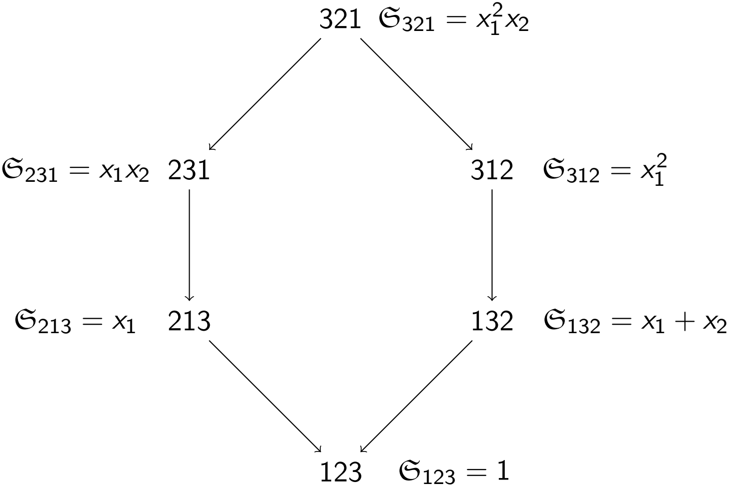
$$\begin{cases} \mathfrak{S}_{w_0} = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}^1 x_n^0 & w_0 = n \ldots 21 \\ \mathfrak{S}_{w_1 \ldots w_{i+1} w_i \ldots w_n} = \partial_i \mathfrak{S}_{w_1 \ldots w_i w_{i+1} \ldots w_n} & w_i > w_{i+1}. \end{cases}$$

The *divided difference operator* ∂_i is

$$\partial_i f = \frac{f(x_1, \ldots, x_i, x_{i+1}, \ldots x_n) - f(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n)}{x_i - x_{i+1}}.$$

Schubert polynomials

Ex: $n = 3$.



Schubert polynomials

Thm: [Lascoux-Schützenberger] If $w \in S_n$, then in $H^\bullet(\mathcal{F}\ell(n))$ we have $[\overline{X_w}] = \mathfrak{S}_{w \cdot w_0}$.

Cor: $\{\mathfrak{S}_w \,:\, w \in S_n\}$ descends to a $\mathbb{Z}$-basis of R_n.

- (Stability) If $w \in S_n$, then $w \times 1 \in S_{n+1}$ (e.g. $231 \times 1 = 2314$).
$$\mathfrak{S}_{w \times 1} = \mathfrak{S}_w$$
- (Positivity) If $u, v \in S_n$, then inside R_n:
$$\mathfrak{S}_u \cdot \mathfrak{S}_v = \sum_{w \in S_n} c_{u,v}^w \cdot \mathfrak{S}_w \quad (c_{u,v}^w \geq 0)$$

Fubini words

Def: A word $w_1 w_2 \ldots w_n$ is *Fubini* if whenever $i > 1$ appears as a letter, so does $i - 1$.

$\mathcal{W}_{n,k} :=$ {length n Fubini words $w_1 \ldots w_n$ with maximum letter k}

$$|\mathcal{W}_{n,k}| = k! \cdot \mathrm{Stir}(n, k).$$

S_n acts on $\mathcal{W}_{n,k}$:

$$\sigma.(w_1 \ldots w_n) = w_{\sigma(1)} \ldots w_{\sigma(n)}.$$

Generalized Coinvariant Algebra

Defn: [HRS] For $k \leq n$, $I_{n,k} \subseteq \mathbb{Z}[\mathbf{x}_n]$ is the ideal

$$I_{n,k} := \langle e_n, e_{n-1}, \ldots, e_{n-k+1}, x_1^k, x_2^k, \ldots, x_n^k \rangle.$$

The ring $R_{n,k}$ is the corresponding quotient.

$$R_{n,k} = \mathbb{Z}[\mathbf{x}_n] / I_{n,k}$$

- $R_{n,k}$ is a graded S_n-module.
- $R_{n,1} = \frac{\mathbb{Z}[\mathbf{x}_n]}{\langle x_1, x_2, \ldots, x_n \rangle} \cong \mathbb{Z}$.
- $I_{n,n} = I_n$ and $R_{n,n} = R_n$.

Thm: [HRS, PR] We have $\mathrm{rank}(R_{n,k}) = |\mathcal{OP}_{n,k}| = k! \cdot \mathrm{Stir}(n,k)$.

Hilbert series

- $[k]_q := 1 + q + \cdots + q^{k-1}$
- $[k]!_q := [k]_q[k-1]_q \cdots [1]_q$

$\mathsf{Stir}_q(0,k) = \delta_{0,k}$ and

$$\mathsf{Stir}_q(n,k) = \mathsf{Stir}_q(n-1,k-1) + [k]_q \cdot \mathsf{Stir}_q(n-1,k).$$

Thm: [HRS, PR] We have $\mathrm{Hilb}(R_{n,k};q) = \mathrm{rev}_q([k]!_q \cdot \mathsf{Stir}_q(n,k))$.

Frobenius series

Thm: [HRS] The *ungraded* S_n-structure of $R_{n,k}$ is

$$R_{n,k} \cong \mathbb{Q}[\mathcal{W}_{n,k}].$$

Thm: [HRS] The *graded* S_n-structure of $R_{n,k}$ is

$$\mathrm{grFrob}(R_{n,k}; q) = \sum_{T \in \mathrm{SYT}(n)} q^{\mathrm{maj}(T)} \cdot \begin{bmatrix} n - \mathrm{des}(T) - 1 \\ n - k \end{bmatrix}_q \cdot s_{\mathrm{shape}(T)}.$$

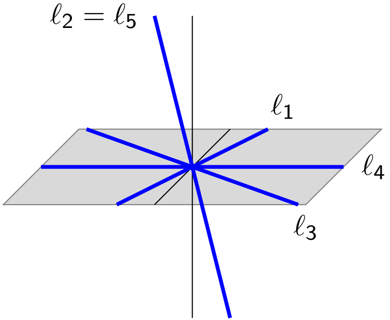
Also equals $(\mathrm{rev}_q \circ \omega)\Delta'_{e_{k-1}} e_n|_{t=0}$.

Q: Geometric model? Don't always have palindromicity: $\mathrm{Hilb}(R_{3,2}; q) = 1 + 3q + 2q^2$.

Spanning Lines

Def: [PR] For $k \leq n$, let $X_{n,k}$ be the space of *line configurations*

$$X_{n,k} := \{(\ell_1, \ldots, \ell_n) : \ell_i \text{ a line in } \mathbb{C}^k \text{ and } \ell_1 + \cdots + \ell_n = \mathbb{C}^k\}.$$



$\in X_{5,3}$

Rmk: For $k = n$, this is *homotopy equivalent* to $\mathcal{F}\ell(n)$.

Spanning Lines

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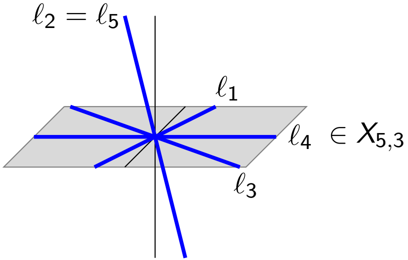

Fact: Over $\mathbb{F}_q$,

$$|X_{n,k}| = q^{\binom{k}{2}} \cdot [k]!_q \cdot \mathrm{Stir}_q(n,k).$$

Spanning Lines

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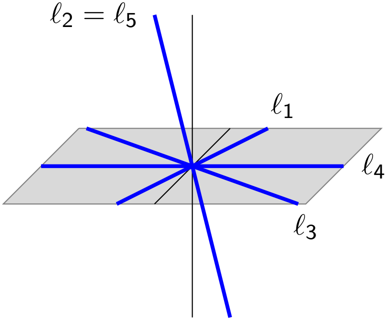


$\in X_{5,3}$

Thm: [PR] $H^{\bullet}(X_{n,k}) = R_{n,k}$. Here

$$x_i \leftrightarrow c_1(\ell_i^*) \in H^2(X_{n,k}).$$

Representation Stability

Def: Let λ be a partition. If $n \geq |\lambda| + \lambda_1$, the *padded partition* is

$$\lambda[n] := (n - |\lambda|, \lambda_1, \lambda_2, \dots)$$

- *Any* partition $\mu \vdash n$ can be written as $\mu = \lambda[n]$ for a unique partition λ.

Def: Suppose $(M_n)_{n \geq 1}$ is a sequence of S_n-modules. Write

$$\mathrm{Frob}(M_n) = \sum_{\lambda} c_{\lambda,n} s_{\lambda[n]}$$

(M_n) is *representation stable* if for all λ, the sequence $c_{\lambda,n}$ is eventually constant.

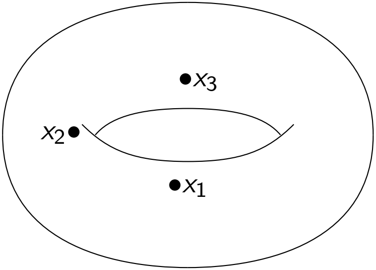
Example: $\mathrm{Frob}((R_n)_3)$

$$
\begin{aligned}
n = 1, 2 &: 0 \\
n = 3 &: s_{111} \\
n = 4 &: s_{211} + s_{31} \\
n = 5 &: s_{311} + s_{41} + s_{32} \\
n = 6 &: s_{411} + s_{51} + s_{42} + s_{33} \\
n = 7 &: s_{511} + s_{61} + s_{52} + s_{43} \\
n = 8 &: s_{611} + s_{71} + s_{62} + s_{53} \\
&\vdots
\end{aligned}
$$

Sources of Stability

Def: The n^{th} *configuration space* of a space X is

$$\mathrm{Conf}_n(X) = \{(x_1, \ldots, x_n) \,:\, x_i \in X,\ x_i \neq x_j \text{ for } i \neq j\}.$$



$\in \mathrm{Conf}_3(\mathrm{torus})$

Meta-Thm: If X is 'nice' then for fixed d, the sequence

$$(H_d(\mathrm{Conf}_n(X); \mathbb{Q}))_{n \geq 1}$$

exhibits representation stability.

Stability for Line Configurations

Def: [PR] For $k \leq n$, let $X_{n,k}$ be the space of 'line configurations'

$$X_{n,k} = \{(\ell_1, \ldots, \ell_n) : \ell_1 + \cdots + \ell_n = \mathbb{C}^k\}.$$

So $H^\bullet(X_{n,k})$ is an S_n-module.

Fact: [PR] For fixed d, either of the sequences

$$\ldots, H^d(X_{n-1,k-1}),\ H^d(X_{n,k}),\ H^d(X_{n+1,k+1}), \ldots$$
$$\ldots, H^d(X_{n-1,k}),\ H^d(X_{n,k}),\ H^d(X_{n+1,k}), \ldots$$

exhibits *representation stability*.

Q: Is there a *geometric* proof?

Structure of $X_{n,k}$

Def: [PR] $X_{n,k} = \{(\ell_1, \ldots, \ell_n) : \ell_1 + \cdots + \ell_n = \mathbb{C}^k\}$.

Thm: [PR] $X_{n,k}$ has a *paving by affines* with cells C_w indexed by *Fubini words* $w = w_1 \ldots w_n \in \mathcal{W}_{n,k}$.

Ex: $(n, k) = (7, 3)$

$$w = 2331231 \leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ \star & 1 & 0 \\ \star & \star & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & \star & \star & 0 & 1 & \star & \star \\ 0 & 1 & 1 & 0 & 0 & 1 & \star \end{pmatrix} = C_w.$$

Word Schubert Polynomials

Q: If $w = w_1 \dots w_n \in [k]^n$ is a Fubini word, what is the class $[\overline{C}_w] \in H^\bullet(X_{n,k}) = R^{\mathbb{Z}}_{n,k}$?

$$\begin{aligned} w &= 2331231 \in [3]^7 \\ \mathrm{conv}(w) &= 2233311 \\ \mathrm{st}(\mathrm{conv}(w)) &= 2435617 \in S_7 \\ \sigma(w) &= 1523647 \in S_7 \end{aligned}$$

Thm: [PR] The class $[\overline{C}_w]$ is represented by

$$\mathfrak{S}_w := \sigma(w)^{-1}.\mathfrak{S}_{\mathrm{st}(\mathrm{conv}(w))} \in \mathbb{Z}[\mathbf{x}_n].$$

Cor: [PR] $\{\mathfrak{S}_w \,:\, w \in [k]^n \text{ Fubini}\}$ descends to a basis for $R_{n,k}$.

Stability

Two ways to grow a Fubini word $w = w_1 \dots w_n \in [k]^n$:

$$1 \times 133213 = 144324$$

$$133213 \circledast 1 = 1332132.$$

Fact: [PR]

- $\mathfrak{S}_{w \circledast 1}(\mathbf{x}_n) = \mathfrak{S}_w(\mathbf{x}_n)$
- $\mathfrak{S}_{1 \times w}(\mathbf{x}^*_{n+1}) \mid_{x_{n+1}=0} = \mathfrak{S}_w(\mathbf{x}^*_n)$

$$\lim_{m \to \infty} \mathfrak{S}_{1^m \times w}(\mathbf{x}_{n+m}) = F_{\mathrm{st}(\mathrm{conv}(w))}$$

(Stanley symmetric function)

- Structure constants not positive :-(

Variation: r-Stirling words

A Fubini word $w_1 \dots w_n$ is *r-Stirling* if its first r letters are distinct.

$$\mathcal{W}_{n,k}^{(r)} = \left\{ \begin{array}{c} r\text{-Stirling Fubini words} \\ \text{of length } n \\ \text{with maximum letter } k \end{array} \right\}.$$

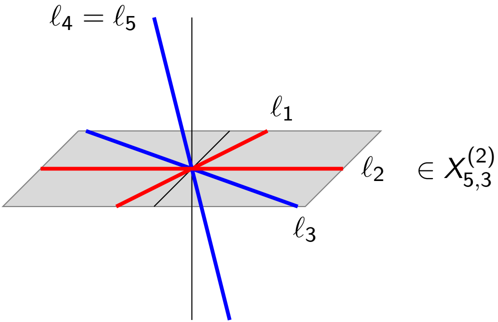
$$\color{red}{241}\color{black}{24132} \in \mathcal{W}_{8,4}^{(3)}$$

Remmel: Is there a version of this for r-Stirling words?

r-Stirling configurations (with A. T. Wilson)

Def: [RW] Let $X_{n,k}^{(r)}$ be the family of line configurations $(\ell_1, \ldots, \ell_n) \in \mathbb{C}^k$ such that

- $\ell_1 + \cdots + \ell_n = \mathbb{C}^k$, and
- $\ell_1, \ldots, \ell_r$ are linearly independent.



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Thm: [RW] Let $r \leq k \leq n$.

- $X_{n,k}^{(r)}$ admits an affine paving indexed by $\mathcal{W}_{n,k}^{(r)}$.
- As *ungraded* $S_r \times S_{n-r}$-modules,

$$H^\bullet(X_{n,k}^{(r)}; \mathbb{Q}) \cong \mathbb{Q}[\mathcal{W}_{n,k}^{(r)}].$$

- As *graded* $S_r \times S_{n-r}$-modules,

$$H^\bullet(X_{n,k}^{(r)}; \mathbb{Q}) = R_{n,k}^{(r)} := \mathbb{Z}[\mathbf{x}_n]/I$$

where I is generated by

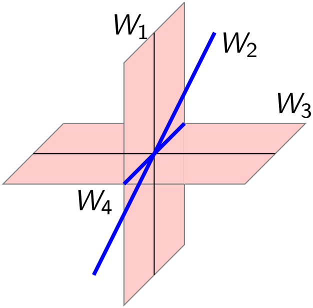
$$x_1^k, \dots, x_n^k, e_n(\mathbf{x}_n), \dots, e_{n-k+1}(\mathbf{x}_n), h_{k-r+1}(\mathbf{x}_r), \dots, h_k(\mathbf{x}_r)$$

Spanning Subspace Configurations

F. Bergeron: What about higher-dimensional subspaces?

Def: A sequence of subspaces $(W_1, \ldots, W_m)$ of $V = \mathbb{C}^k$ is a *spanning configuration* if $W_1 + \cdots + W_m = V$. If $\alpha = (\alpha_1, \ldots, \alpha_m)$ is a composition,

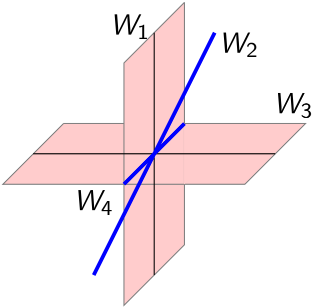
$$X_{\alpha,k} := \{\text{spanning configurations in } \mathbb{C}^k \text{ with dimension vector } \alpha\}.$$



$\in X_{(2,1,2,1),3}$

Spanning Subspace Configurations

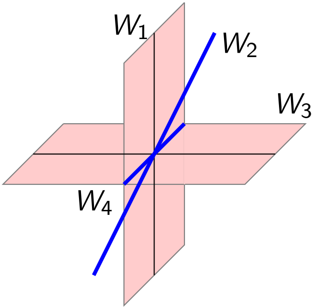
$X_{\alpha,k} :=$ {spanning configurations in $\mathbb{C}^k$ with dimension vector α}.



$\in X_{(2,1,2,1),3}$

- $\alpha = (1, \dots, 1) \Rightarrow$ recover $X_{n,k}$.
- $\alpha_i = k$ for some $i \Rightarrow Gr(\alpha_1, k) \times Gr(\alpha_2, k) \times \cdots$ (Grassmannian product)
- $\sum_i \alpha_i = k \Rightarrow \mathcal{F}\ell(\alpha_1, \alpha_1 + \alpha_2, \dots)$ (partial flag variety)

Spanning Subspace Configurations



$\in X_{(2,1,2,1),3}$

Thm: [PR] $H^{\bullet}(X_{\alpha,k})$ is a free $\mathbb{Z}$-module, rank $=$ number of k row 0,1-matrices with column sums α and no zero rows.

$$\text{if } \alpha = (2,1,2,1) \text{ and } k = 3, \text{ contributor is } \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

Spanning Subspace Configurations

Thm: [PR] Let $\alpha = (\alpha_1, \ldots, \alpha_m) \in [k]^m$ and $n := \alpha_1 + \cdots + \alpha_m$. Then

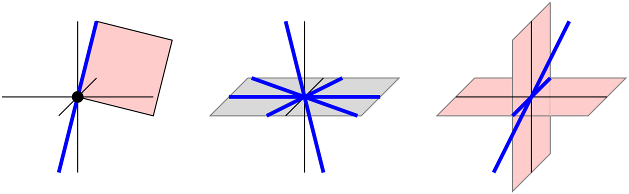
$$H^\bullet(X_{\alpha,k}) = (\mathbb{Z}[\mathbf{x}_n]/I)^{S_\alpha},$$

where I is generated by

- $e_n(\mathbf{x}_n), e_{n-1}(\mathbf{x}_n), \ldots, e_{n-k+1}(\mathbf{x}_n)$, and
- $h_{k-\alpha_i+1}, h_{k-\alpha_i+2}, \ldots, h_k$ in $\{x_{\alpha_1+\cdots+\alpha_{i-1}+1}, \ldots, x_{\alpha_1+\cdots+\alpha_i}\}$ for $i = 1, 2, \ldots, m$.

Here $x_1, \ldots, x_n$ are Chern roots of $W_1^* \oplus \cdots \oplus W_m^*$.

Rmk: Presentation of cohomology relies on a *Demazure character dual Pieri rule* of Haglund-Luoto-Mason-van Willigenburg.



Thanks for listening!

- J. Haglund, B. Rhoades, and M. Shimozono. Ordered set partitions, generalized coinvariant algebras, and the Delta Conjecture. *Adv. Math.*, **329** (2018) 851–915.
- B. Pawlowski and B. Rhoades. A flag variety for the Delta Conjecture. `arXiv:1711.08301`
- B. Rhoades and A. T. Wilson. Line configurations and r-Stirling partitions. *J. Comb.*, to appear.