Ultraproduct embeddings and amenability for tracial von Neumann algebras

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Preliminaries

Definition

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Let (N, τ) and (M, σ) be two tracial von Neumann algebras. A unital $*$-homomorphism $\pi : N \to M$ is an *embedding* if it is injective and $\sigma \circ \pi = \tau$.

Preliminaries

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A tracial von Neumann algebra (N, τ) is *hyperfinite* if it is the σ-weak closure of an increasing union of finite-dimensional subalgebras. Let R denote the separably acting hyperfinite II_1-factor. This is equivalent to amenable, injective, and semidiscrete.

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Proposition

If (N, τ) is a separable amenable tracial von Neumann algebra, let $\{M_k\}$ be a sequence of II_1-factors, and let $\mathcal{U}$ denote a free ultrafilter on $\mathbb{N}$, then any two embeddings $\pi, \rho : N \to \prod_{k\to\mathcal{U}} M_k$ are unitarily conjugate.

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Definition

A tracial von Neumann algebra (N, τ) *satisfies the Connes Embedding Problem (CEP)* if there is an embedding $\pi : N \to R^{\mathcal{U}}$.

Jung's characterization

Theorem (Jung, '07)

Let (N, τ) be a tracial von Neumann algebra satisfying the CEP. Then N is amenable if and only if any two embeddings $\pi, \rho : N \to R^{\mathcal{U}}$ are unitarily conjugate.

Sketch of Jung's argument

Definition

Let $X = \{x_1, \ldots, x_n\} \subset (N)_1^{\text{s.a.}}$. The n-tuple X is *tubular* if for every $\varepsilon > 0$ there is a $\delta > 0$ and $m \in \mathbb{N}$ such that for any $J \in \mathbb{N}$ and $\xi, \eta \in \Gamma(X; m, J, \delta)$ there is a unitary $u \in \mathcal{U}(\mathbb{M}_J)$ such that

$$||\xi - u^*\eta u||_2 < \varepsilon.$$

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Lemma

Let (N, τ) be a tracial von Neumann algebra satisfying the CEP with $N = W^*(X)$. If any two embeddings $\pi, \rho : N \to R^{\mathcal{U}}$ are unitarily conjugate, then X is (quasi-)tubular.

Sketch of Jung's argument

Lemma

Let (N, τ) be a tracial von Neumann algebra satisfying the CEP with $N = W^*(X)$. If X is (quasi-)tubular, then for any $\varepsilon > 0$ there is an embedding $\pi : N \to R^{\mathcal{U}}$ and a finite dimensional subalgebra $A \subset R^{\mathcal{U}}$ such that $\pi(X) \subset_{\varepsilon, ||\cdot||_2} A$.

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Proposition

Let (N,τ) be a tracial von Neumann algebra satisfying the CEP with $N = W^*(X)$. If for any $\varepsilon > 0$ and there is an embedding $\pi : N \to R^{\mathcal{U}}$ and a finite dimensional subalgebra $A \subset R^{\mathcal{U}}$ such that $\pi(X) \subset_{\varepsilon,||\cdot||_2} A$, then N is semidiscrete.

ucp conjugation

Definition

Let (N, τ) be a separable tracial von Neumann algebra satisfying CEP. Two embeddings $\pi, \rho : N \to R^{\mathcal{U}}$ are *ucp-conjugate* if there is a sequence of ucp maps $\varphi_k : R \to R$ such that $\pi = (\varphi_k)_{\mathcal{U}} \circ \rho$. That is, for every $x \in N$, if $\rho(x) = (a_k)_{\mathcal{U}}$ then $\pi(x) = (\varphi_k(a_k))_{\mathcal{U}}$.

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Theorem (A.-Kunnawalkam Elayavalli, '19)

Let (N, τ) be a separable tracial von Neumann algebra satisfying CEP. Then N is amenable if and only if any two embeddings $\pi, \rho : N \to R^{\mathcal{U}}$ are ucp-conjugate.

Sketch of argument

Definition

The n-tuple $X \subset (N)_1^{\text{s.a.}}$ is *completely tubular* if for every $\varepsilon > 0$ there is a $\delta > 0$ and $m \in \mathbb{N}$ such that for any $J \in \mathbb{N}$ and $\xi, \eta \in \Gamma(X; m, J, \delta)$ there is a ucp map $\varphi : \mathbb{M}_J \to \mathbb{M}_J$ such that

$$||\xi - \varphi(\eta)||_2 < \varepsilon.$$

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Lemma

Let (N, τ) be a tracial von Neumann algebra satisfying the CEP with $N = W^*(X)$. If any two embeddings $\pi, \rho : N \to R^{\mathcal{U}}$ are ucp-conjugate, then X is completely tubular.

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Lemma

Let (N, τ) be a tracial von Neumann algebra satisfying the CEP with $N = W^*(X)$. If X is completely tubular, then for any $\varepsilon > 0$ and any finite subset $F \subset (N)_1$ there is a $J \in \mathbb{N}$ and a ucp map $\varphi : \mathbb{M}_J \to N$ such that $F \subset_{\varepsilon, ||\cdot||_2} \varphi((\mathbb{M}_J)_1)$.

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Proposition (Kishimoto, unpublished)

Let (N, τ) be a separable tracial von Neumann algebra. Then N is injective if and only if for any $\varepsilon > 0$ and any finite subset $F \subset (N)_1$ there is a $J \in \mathbb{N}$ and a ucp map $\varphi : \mathbb{M}_J \to N$ such that $F \subset_{\varepsilon, ||\cdot||_2} \varphi((\mathbb{M}_J)_1)$.

Consequences

Corollary (A.-Kunnawalkam Elayavalli, '19)

Let (N, τ) be a separable tracial von Neumann algebra satisfying the CEP, and for each $k \in \mathbb{N}$ let M_k be a II_1-factor. Then N is amenable if and only if any two embeddings $\pi, \rho : N \to \prod_{k \to \mathcal{U}} M_k$ are (unitarily /ucp-)conjugate.

Ozawa's improvement

Let $\mathbb{H}\text{om}(N, M)$ denote the space of all unital $*$-homomorphisms of N into M modulo unitary equivalence. Endow $\mathbb{H}\text{om}(N, M)$ with the natural topology of point-$||\cdot||_2$ convergence of representatives. Jung showed that N is amenable if and only if $|\mathbb{H}\text{om}(N, R^{\mathcal{U}})| = 1$.

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Theorem (Ozawa '11)

Let (N, τ) be a separable tracial von Neumann algebra satisfying CEP. Then N is amenable if and only if $\mathbb{H}\mathrm{om}(N, R^{\mathcal{U}})$ is separable.

Popa's question

Let $\{M_k\}$ be a sequence of II_1-factors. The above consequence says that N is amenable if and only if $|\mathbb{H}\text{om}(N, \prod_{k\to\mathcal{U}} M_k)| = 1$.

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Tracial stability

Definition

Let $\mathscr{C}$ denote a class of C^*-algebras. A C^*-algebra $\mathcal{A}$ is $\mathscr{C}$*-tracially stable* if for every unital $*$-homomorphism $\pi : \mathcal{A} \to \prod_{k\to\mathcal{U}}(\mathcal{A}_k, \tau_k)$ with $\mathcal{A}_k \in \mathscr{C}$ there is a sequence of unital $*$-homomorphisms $\pi_k : \mathcal{A} \to \mathcal{A}_k$ such that $\pi(a) = (\pi_k(a))_{\mathcal{U}}$ for every $a \in \mathcal{A}$.

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Example

R is $\mathbf{II}_1$-tracially stable. In fact, R is the only $\mathbf{II}_1$-tracially stable II_1-factor.

Self-tracial stability

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Let (N, τ) be a separable tracial von Neumann algebra satisfying the CEP. Then N is amenable if and only if N is self-tracially stable.

Corollary (A.-Kunnawalkam Elayavalli, '19)

If (N, τ) is self-tracially stable and non-amenable, then N does not satisfy the CEP.

THANKS!

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