

# Classification Problems in Von Neumann Algebras

Adrian Ioana (University of California, San Diego),  
Jesse Peterson (Vanderbilt University)

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## 1 Overview of the Field

### 1.1 Overview of the topic of the workshop

Von Neumann algebras are algebras of bounded operators on a Hilbert space that are closed under taking the adjoint and in the weak operator topology. If their center consists only of multiples of the identity, they are called factors. Since its early development in the 1930's, the subject has been closely connected to ergodic theory and group theory, via a seminal construction which associates von Neumann algebras to countable groups and their actions on measure spaces. A major theme in the area remains to classify such group and group measure space von Neumann algebras. This problem is typically studied when the von Neumann algebras are  $\text{II}_1$  factors, which corresponds to the groups being infinite conjugacy class (icc) and respectively the actions being free, ergodic and probability measure preserving. Early work, concentrating on the amenable case, culminated with Connes' celebrated classification of amenable factors in the 1970's [8]. By contrast, the non-amenable case turned out to be extremely difficult, in large part remaining intractable.

In recent years, there have been spectacular advances in the classification of von Neumann algebras, generated largely by Popa's discovery of deformation/rigidity theory in the early 2000s [31]. Much of this progress has been stimulated by the reemerging connections to ergodic theory and group theory, as well as recent newly developed connections to measured group theory and logic. We next overview some of the recent breakthroughs made on classification problems within von Neumann algebras as well as on problems at the interface with measured group theory and model theory.

#### 1.1.1 Group von Neumann algebras

First, remarkable progress has been made in the study of structural properties and the classification of group von Neumann algebras. By using deformation/rigidity theory, Ozawa and Popa (2007) proved that the free group factors  $L(\mathbb{F}_n)$ , with  $n \geq 2$ , satisfy a structural property called strong solidity [27]. This considerably strengthened the absence of Cartan subalgebras for such factors obtained by Voiculescu in the mid 1990s using free probability theory [34]. A surprising result of Guionnet and Shlyakhtenko (2012) shows that for small  $q$  and a fixed number of generators, the  $q$ -deformed free group factors are all isomorphic [19]. Despite this progress in understanding the structure of the free group factors, it is a longstanding open problem whether  $\text{II}_1$  factors arising from non-abelian free groups of different ranks are isomorphic or not.

Another outstanding problem asks to prove that the arithmetic groups  $\text{PSL}_k(\mathbb{Z})$  give rise to non-isomorphic  $\text{II}_1$  factors, for different values of  $k$ . More generally, a far-reaching rigidity conjecture of Connes from 1980

predicts that property (T) groups having isomorphic von Neumann factors must be isomorphic. Since property (T) passes from groups to their von Neumann algebras, the conjecture is equivalent to a superrigidity statement: the von Neumann algebra of a property (T) group completely determines the group. A breakthrough in this direction was made by Ioana, Popa and Vaes who discovered the first examples of superrigid groups [23]. However, these groups do not have property (T), leaving open the problem of finding even a single example of an icc superrigid group with property (T). In 2018, using a new notion for groups called proper proximality, the first structural results for the von Neumann algebras of  $\mathrm{PSL}_k(\mathbb{Z})$  with  $k \geq 3$  were obtained in [6]. This opens up a promising angle of attack on the isomorphism problem for  $L(\mathrm{PSL}_k(\mathbb{Z}))$ .

### 1.1.2 Group measure space von Neumann algebras

Secondly, some of the most striking recent developments in the theory of von Neumann algebras concern the classification of group measure space von Neumann algebras. These include the discovery by Peterson [28], Popa and Vaes [32], and Ioana [21] of the first families of group actions which are  $W^*$ -superrigid, in the sense that the group and action can be entirely recovered from their von Neumann algebras. Another major accomplishment is due to Popa and Vaes (2012) who showed that group measure space factors arising from arbitrary free actions of free groups have a unique Cartan subalgebra [33] up to unitary conjugacy. In combination with work of Gaboriau ([16],[17]), they deduced that factors associated to actions of free groups of different ranks are never isomorphic. This settled the group measure space version of the free group factor problem. More generally, a well-known conjecture predicts that if a group  $G$  has a positive first  $L^2$ -Betti number, then any  $\mathrm{II}_1$  factor associated to a free action of  $G$  must have a unique Cartan subalgebra, up to unitary conjugacy. If true, it would follow that the first  $L^2$ -Betti number of the group is an isomorphism invariant of the group measure space factor.

### 1.1.3 Von Neumann algebras and measured group theory

Thirdly, recently there have been important advances on several classification problems at the interface of von Neumann algebras and measured group theory. The latter studies groups by looking at the orbit structure of their measure preserving actions on probability spaces. A central theme is to classify such actions up to orbit equivalence. This is closely related to the classification of group measure space von Neumann algebras. Much of the recent progress on these classification problems has been achieved hand-in-hand. For instance, ideas from the theory of von Neumann algebras were used to show that any non-amenable group admits uncountably many free ergodic actions that are not orbit equivalent [11] (see also [20] and [18]). Using his deformation/rigidity theory, Popa famously proved that Bernoulli actions of property (T) groups are superrigid with respect to orbit equivalence [30]. Another theme in measured group theory which has received considerable attention in the last few years is the classification of conjugation invariant positive definite functions (in short, characters) of a given group. This is equivalent to classifying all  $\mathrm{II}_1$  factor representations of the group. In 2007, Bekka proved that  $\mathrm{PSL}_k(\mathbb{Z})$  with  $k \geq 3$  has only one such representation, the left regular representation [3]. In 2015, Peterson obtained a far-reaching generalization which shows the same holds for all lattices in higher rank simple Lie groups [29]. This confirmed a conjecture of Connes from the 1980s, and extended Margulis' celebrated normal subgroup theorem.

### 1.1.4 Von Neumann algebras and model theory

Finally, there has been significant interest recently in the model theory of von Neumann algebras initiated by Farah, Hart and Sherman ([13], [14], [15]). A primary goal here is to classify  $\mathrm{II}_1$  factors up to elementary equivalence. Two  $\mathrm{II}_1$  factors are elementarily equivalent if and only if they admit isomorphic ultrapowers, with respect to some, possibly very large, ultrafilters. Until recently, only three distinct elementary classes of  $\mathrm{II}_1$  factors have been known, similar to the isomorphism situation for separable  $\mathrm{II}_1$  factors in the 1940s-1960s. However, Boutonnet, Chifan and Ioana succeeded in providing a continuum of non-elementarily equivalent  $\mathrm{II}_1$  factors [2]. Elementary equivalence of  $\mathrm{II}_1$  factors still remains largely mysterious. A basic question is to find non-elementary  $\mathrm{II}_1$  factors that are not McDuff, or perhaps do not even have property Gamma.

## 1.2 Objectives and structure of the workshop

The principal aim of this workshop was to further research on the classification of von Neumann algebras, by capitalizing on the recent advances described above and taking advantage of newly developed connections to other areas. The workshop brought together leading experts and young researchers working in von Neumann algebras and interacting areas, such as measured group theory, logic,  $C^*$ -algebras and free probability.

The workshop had 37 participants in a very active area at the intersection of operator algebras with ergodic theory and logic. There were 25 talks, of 50 or 25 minutes each, about one third of which were given by promising young reseachers (seven postdoctoral researchers and one graduate student). The talks were generally of high quality and generated a lot of stimulating discussions and interactions among the participants.

## 2 Presentation highlights

In this section, we discuss some of the main results presented, divided into several thematic directions:

### 2.1 Structure and classification of von Neumann algebras

Several talks were devoted to problems concerning the structure and classification of von Neumann algebras that arise intrinsically in the theory.

#### 2.1.1 Marius Junge: $q$ -Gaussian von Neumann algebras

Junge discussed a new class of von Neumann algebras, called  $q$ -Gaussian algebras with coefficients, which are  $q$ -Gaussian analogues of amalgamated free products. He presented a strong solidity theorem for such algebras inspired by Popa and Vaes' work [33] discussed in Section 1.1.2.

#### 2.1.2 Yusuke Isono: Popa's intertwining theory for type III factors

A fundamental tool in deformation/rigidity theory is Popa's intertwining theorem. This provides a method to prove unitary conjugacy of subalgebras of a given  $\text{II}_1$  factor. In order to apply deformation/rigidity theory to von Neumann algebras of type III, there have been several attempts to generalize the intertwining technique to this setting. Isono explained his very recent work which extends the intertwining technnique to general von Neumann algebras of type III [25].

Two of the talks were devoted to superrigidity phenomena for group algebras, see Section 1.1.1. We recall that a countable group  $\Gamma$  is called  $W^*$ -superrigid (respectively,  $C^*$ -superrigid) if it can be recovered from its group von Neumann algebra  $L(\Gamma)$  (respectively, reduced  $C^*$ -algebra  $C_r^*(\Gamma)$ ).

#### 2.1.3 Sven Raum: Superrigidity for group operator algebras

Raum started with an excellent introduction to the state-of-art of the  $C^*$ -superrigidity problem for groups. He then presented a recent result showing that any torsion free 2-step nilpotent group is  $C^*$ -superrigid [12].

#### 2.1.4 Ionut Chifan: Some rigidity aspects in von Neumann algebras and $C^*$ -algebras arising from groups.

In a related talk, Chifan explained how techniques used to establish  $W^*$ -superrigidity results in [23] can be employed to provide a new class of groups which are  $C^*$ -superrigid. These groups appear as wreath products with non-amenable core. In particular, they are non-amenable and thus different than the groups considered by Raum. Moreover, for these groups one can calculate the automorphism group of their reduced  $C^*$ -algebras.

### 2.1.5 Ben Hayes: Maximal rigid subalgebras of deformations and $L^2$ -cohomology, I, and Rolando de Santiago: Maximal rigid subalgebras of deformations and $L^2$ -cohomology, II

Hayes and de Santiago gave two back-to-back talks on their recent joint work with Hoff and Sinclair [10]. Hayes presented a theorem which implies that if  $G$  is a group with positive first  $L^2$ -Betti number, then its von Neumann algebra  $L(G)$  cannot be generated by two subalgebras with property (T) with diffuse intersection. This is a consequence of a conceptual result showing that if a diffuse subalgebra  $Q$  is rigid (in the sense of Popa) with respect to a mixing malleable deformation of  $M$ , then it is contained in a unique maximal rigid subalgebra of  $M$ . In the second talk, de Santiago presented additional consequences of these techniques, including a result asserting that the quasi-normalizer of a rigid diffuse subalgebra generates a rigid subalgebra.

### 2.1.6 Lauren Ruth: Von Neumann equivalence and properly proximal groups.

Ruth explained an interesting new notion of equivalence for countable groups, called von Neumann equivalence, which generalizes both  $W^*$ -equivalence and measure equivalence. Solving a problem posed in [6], Ruth and co-authors proved that proper proximality is invariant under von Neumann equivalence, and thus under measure equivalence [22].

### 2.1.7 David Jekel: Free complementation of certain MASAs in $L(\mathbb{F}_d)$ via conditional transport of measure

Jekel discussed recent exciting work in which he establishes a “triangular” version of the free transport phenomenon pioneered by Guionnet and Shlyakhtenko in [19] (see Section 1.1.1). This has striking applications to the structure of the free groups factors  $L(\mathbb{F}_d)$  which can be realized as the von Neumann algebra  $W^*(S_1, \dots, S_d)$  generated by  $d$  freely independent semicircular elements. Specifically, Jekel shows that if  $P$  is any non-commutative polynomial, then for every small enough  $\varepsilon > 0$ , the von Neumann algebra generated by  $S_1 + \varepsilon \cdot P(S_1, \dots, S_d)$  is freely complemented, and thus maximal amenable, in  $L(\mathbb{F}_d)$ .

## 2.2 Von Neumann algebras and measured group theory

Several talks were focused on the topics at the intersection of von Neumann algebras and measured group theory discussed in Section 1.1.3. First, three talks were devoted the study of orbit equivalence relations associated to probability measure preserving actions of countable groups.

### 2.2.1 Yoshikata Kida: Groups with infinite FC-center have the Schmidt property

A countable group is said to have the Schmidt property if it admits an ergodic free probability measure preserving action such that the full group of the associated orbit equivalence relation contains a non-trivial central sequence. Klaus Schmidt asked whether any inner amenable group has this property. Kida explained a recent result showing that any countable group with infinite FC-center has the Schmidt property [26].

### 2.2.2 Pieter Spaas: The Jones-Schmidt property and central sequence algebras

Spaas presented examples of countable equivalence relations that do not have the Jones-Schmidt property, answering a question of Vaughan Jones and Klaus Schmidt from 1985 [24]. He also discussed connections with structural properties of central sequence algebras, and implications for unique McDuff decompositions.

### 2.2.3 Daniel Drimbe: Orbit equivalence rigidity for product actions

Drimbe presented a new type of a rigidity phenomenon in orbit equivalence: for a large class of product actions  $\Gamma = \Gamma_1 \times \dots \times \Gamma_n$  on  $X = X_1 \times \dots \times X_n$ , the orbit equivalence relation remembers the product structure of the group action [9]. More precisely, if a free ergodic action of a group  $\Lambda$  on  $Y$  is orbit equivalent to  $\Gamma$  on  $X$ , then essentially  $\Lambda$  is a product of  $n$  groups such that its action on  $Y$  is a product of  $n$  actions.

Secondly, we also had two talks devoted to character rigidity.

### 2.2.4 Bachir Bekka: Characters of algebraic groups

Bekka discussed the characters of the group  $G(k)$  of  $k$ -points of an algebraic group  $G$  defined over a field  $k$ . Under the assumption that  $G(k)$  is generated by its unipotent elements, he presented a complete classification of  $\text{Char}(G(k))$  in two cases: 1)  $G$  is  $k$ -simple and  $k$  is an arbitrary infinite field; 2)  $G$  is an arbitrary algebraic group and  $k$  is a number field [4].

### 2.2.5 Remi Boutonnet: Stationary characters on lattices in semi-simple groups

A positive definite function  $\phi$  on a countable group  $G$  is called a stationary character with respect to a probability measure  $\mu$  on  $G$  if it satisfies  $\mu * \phi = \phi$ . Boutonnet presented a recent breakthrough joint with Houdayer in which they show that stationary characters of higher-rank lattices  $G$  are genuine characters [5]. He presented several applications of this result to representation theory, operator algebras and topological dynamics. In particular, he explained a striking consequence which is new even for  $G = \text{SL}_k(\mathbb{Z})$  with  $k \geq 3$ : any weakly mixing unitary representation of  $G$  weakly contains the left regular representation. He also explained how this work allows to recover Peterson's character rigidity theorem discussed in Section 1.1.3.

## 2.3 Von Neumann algebras, the Connes Embedding Problem and Model Theory

The Connes Embedding Problem (CEP) is a major open problem in operator algebras which asks whether any separable  $\text{II}_1$  factor  $M$  can be embedded into the ultrapower  $R^\omega$  of the hyperfinite  $\text{II}_1$  factor  $R$ . While originally formulated in terms of von Neumann algebras, this problem has equivalent formulations in many different areas, including  $C^*$ -algebras, quantum information theory and model theory.

### 2.3.1 Isaac Goldbring: Playing games with $\text{II}_1$ factors

Goldbring introduced several games that one can play with  $\text{II}_1$  factors and the connection between these games and other active areas of research, including the Connes Embedding Problem and the study of elementarily equivalent  $\text{II}_1$  factors. Towards the end of the talk, he mentioned an open question asking whether the hyperfinite  $\text{II}_1$  factor  $R$  is infinitely generic, and pointed out that a negative answer would lead to the first example of two non-elementarily equivalent existentially closed  $\text{II}_1$  factors.

### 2.3.2 Scott Atkinson: Ultraproduct embeddings and amenability for tracial von Neumann algebras

Atkinson discussed a recent result joint with S.Kunnawalkam Elayavalli showing that for Connes embeddable tracial von Neumann algebras  $M$  amenability is characterized by the property that any two embeddings into an ultrapower of  $R$  are conjugate by unital completely positive maps [1]. He also explained a theorem, partially answering a question of Popa, which shows amenability of  $M$  is also equivalent to the separability of the space of embeddings of  $M$  into any ultraproduct of  $\text{II}_1$  factors.

### 2.3.3 Kate Juschenko: Representations of products of the free group, transport operators and Connes' embedding problem

Juschenko discussed several conjectures related to the Connes' embedding problem, in the spirit of Kirchberg's equivalent reformulation in terms of unitary representation of products of free groups.

### 2.3.4 Thomas Sinclair: Tensor products of matrix convex sets

Sinclair spoke on joint work with R. Araiza and A. Dor-On on the tensor theory of matrix convex sets. As an application, they obtain a new formulation of the Connes' embedding problem via noncommutative Choquet theory.

### 2.3.5 Ian Charlesworth: Matrix models for $\varepsilon$ -independence

Charlesworth explained how to construct matrix models (in a certain tensor product of matrix algebras) for the notion of  $\varepsilon$ -independence, which is an interpolation of classical and free independence.

## 2.4 C\*-algebras

### 2.4.1 Tim de Laat: Exotic group C\*-algebras of simple Lie groups with real rank one.

The reduced and full C\*-algebras of a locally compact group  $G$  coincide exactly when  $G$  is amenable. In general, for non-amenable groups  $G$ , there can be many *exotic* C\*-algebras sitting between the reduced and full C\*-algebras. De Laat presented results on exotic C\*-algebras of Lie groups of rank of  $G$  which arise from  $L^p$ -integrable representations of  $G$ .

### 2.4.2 Claire Anantharaman-Delaroche: Weak containment vs amenability for group actions and groupoids

A groupoid or a group action on a C\*-algebra has the weak containment property if the associated reduced and full C\*-algebras coincide. For groups, this property is equivalent to amenability. Anantharaman Delaroche discussed the subtle question of whether in general the weak containment property implies amenability.

### 2.4.3 Alain Valette: Explicit Baum-Connes for $\mathbb{Z}^2 \rtimes \mathbb{F}_2$

Valette reported on an explicit calculation of the assembly map appearing in the Baum-Connes conjecture in the case of the semi-direct product  $G = \mathbb{Z}^2 \rtimes \mathbb{F}_2$ , where  $\mathbb{F}_2 < \mathrm{SL}_2(\mathbb{Z})$  is a free subgroup on 2 generators.

### 2.4.4 Kristin Courtney: Amalgamated Products of Strongly RFD C\*-algebras arising from locally compact groups

Courtney's talk discussed joint work with T. Shulman giving new instances of when residual finite dimensionality of C\*-algebras is preserved under the amalgamated free product construction. This is proven to hold for pairs of "strongly residually finite dimensional" C\*-algebras amalgamated over a central subalgebra.

## 2.5 Other topics

### 2.5.1 Stefaan Vaes: Ergodicity and type of nonsingular Bernoulli actions

Vaes reported on a joint work with M. Björklund and Z. Kosloff on nonsingular Bernoulli actions. In this very impressive work, the authors prove in almost complete generality that a nonsingular Bernoulli action is either dissipative or weakly mixing, and determine its Krieger type.

### 2.5.2 Robin Tucker-Drob: Inner amenability and the location lemma

Tucker-Drob overviewed recent results on inner amenable groups, focusing on joint work with P. Wesolek and B. Duchesne, in which they obtained a complete characterization of inner amenability for generalized wreath product groups.

### 2.5.3 Andrew Marks: Measurable realizations of abstract systems of congruence

An abstract system of congruences describes a way of partitioning a space into finitely many pieces satisfying certain congruence relations. Marks discussed the question of when there are realizations of abstract systems of congruences satisfying various measurability constraints (e.g., requiring the pieces to be Borel). Marks presented a general result showing that, under certain assumptions, abstract systems of congruences can be realized by any hyperfinite action of  $\mathbb{F}_2$  on a standard probability space using measurable pieces.

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