Group action induced Cartan pairs

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Outline

I. Introduction: outline, general motivation

II. Basic Notions: Cartan subalgebras, groupoids

III. Theorems and application

Abelian operator algebras are well-understood:

- von Neumann algebras: L^∞ spaces
- C^*-algebras: $C_0(X)$ for l.c. Hausdorff spaces X.

The study of non-abelian operator algebras is facilitated by examination of abelian subalgebras. In particular:

- Cartan von Neumann subalgebras (Feldman-Moore '77),
- Diagonal C^*-subalgebras (Kumjian, '86),
- Cartan C^*-subalgebras (Renault, '08): correspond to étale, 2^{nd} countable, locally compact Hausdorff, topologically principal twisted groupoids,
- **Γ-Cartan C^*-subalgebras** (Brown-Fuller-Pitts-R, '18): Introduced to generalize Renault's results. Method inspired by work of Ara-Bosa-Hazrat-Sims on Steinberg algebras.

Recall: A C^*-algebra is a norm-closed *-subalgebra of $\mathcal{B}(H)$.

A maximal abelian C*-subalgebra $\mathcal{B} \subseteq \mathcal{A}$ is **Cartan** (Renault) if

- There is a faithful conditional expectation $\mathcal{A} \to \mathcal{B}$,
- $\mathcal{B}$ contains an approximate unit of $\mathcal{A}$, and
- The normalizers of $\mathcal{B}$ in $\mathcal{A}$ generate $\mathcal{A}$.

Example. Let E be a directed graph with $E^* = \{\text{finite paths}\}$.

$$C^*(E) := C^*(\{t_\alpha \,|\, \alpha \in E^*\}) = \overline{\text{span}}\{t_\alpha t_\beta^* \,|\, s(\alpha) = s(\beta)\}$$

with the t_α partial isometries satisfying Cuntz-Krieger relations.

- The diagonal subalgebra $\mathcal{D} := \overline{\text{span}}\{t_\alpha t_\alpha^* \,|\, \alpha \in E^*\}$ is Cartan iff E has no cycles without entry.
- Nagy-R ('12): In any case, the *cycline subalgebra* $\mathcal{M} \supseteq \mathcal{D}$ is Cartan. ($\mathcal{M}$ appears in uniqueness theorems.)
- Brown-Nagy-R-Sims-Williams ('16): The cycline subalgebra of a (higher-rank) k-graph is not necessarily Cartan.

A C^*-algebra $\mathcal{A}$ is *topologically graded* by a group Γ if there are linearly independent subspaces $\mathcal{A}_t$ satisfying the following.

- $\mathcal{A} = \overline{\operatorname{span}}\{\mathcal{A}_t \,|\, t \in \Gamma\}$,
- $\forall s, t \in \Gamma,\ \ \mathcal{A}_t\mathcal{A}_s \subseteq \mathcal{A}_{t+s},\ \ \mathcal{A}_t^* = \mathcal{A}_{-t}$, and
- there is a conditional expectation of $\mathcal{A}$ onto $\mathcal{A}_0$.

Exel: Every topological Γ-grading on a C^*-algebra $\mathcal{A}$ is induced by a strongly continuous action of $\hat{\Gamma}$ on $\mathcal{A}$.

Conversely, a strongly continuous action $\hat{\Gamma} \times \mathcal{A} \to \mathcal{A}$ gives rise to a topological Γ-grading on $\mathcal{A}$. In particular,

$$\mathcal{A}_t = \{a \in \mathcal{A} \,|\, \forall \omega \in \hat{\Gamma}\ \ \omega \cdot a = \langle t, \omega \rangle a\}.$$

Remark: We have $\mathcal{A}_0 = \mathcal{A}^{\hat{\Gamma}}$, the fixed point algebra.

Definition (Brown-Fuller-Pitts-R, '18)

Suppose $\mathcal{A} = \overline{\mathrm{span}}\{\mathcal{A}_t \mid t \in \Gamma\}$ is topologically graded by Γ. We say a C^*-subalgebra $\mathcal{D} \subseteq \mathcal{A}$ is **Γ-Cartan** if

- $\mathcal{D}$ contains an approximate unit for $\mathcal{A}$,
- the normalizers of $\mathcal{D}$ form a dense subset in $\mathcal{A}$, and
- $\mathcal{D} \subseteq \mathcal{A}_0$ is Cartan.

Kumjian ('86): Diagonal pairs correspond to twisted principal groupoids.

Renault ('08): Cartan pairs correspond to twisted topologically principal groupoids.

BFPR ('18): Γ-Cartan pairs correspond to Γ-graded Γ-topologically principal twisted groupoids.

A **groupoid** is a small category G in which every morphism has an inverse. Denote by $G^{(0)}$ the objects (unit space of identity morphisms), and range and source maps $r, s : G \to G^{(0)}$.

The *isotropy subgroupoid* $\mathrm{Iso}(G) = \{g \in G \mid r(g) = s(g)\}$ plays a significant role in the analysis of groupoid C^*-algebras.

A groupoid is *principal* if $G^{(0)} = \mathrm{Iso}(G)$ (no nontrivial loops).

A *topological groupoid* is a groupoid endowed with a topology with respect to which inversion and composition are continuous. When r and s are local homeomorphisms, we say G is *étale*.

A groupoid is *topologically principal* if the points in $G^{(0)}$ with trivial isotropy form a dense set in $\mathrm{Iso}(G)$.

Example: the path groupoid G_E of a directed graph E

$$E^* := \text{ space of (finite) paths; } \quad \ell(\alpha) := \text{ length of } \alpha$$

$$E^\infty := \text{ space of one-sided infinite paths (no source)}$$

$$G_E = \{(\overset{r}{\alpha y},\ \ell(\alpha) - \ell(\beta),\ \overset{s}{\beta y}) \mid y \in E^\infty,\ \alpha, \beta \in E^*\}$$

$$(x, m, y)^{-1} = (y, -m, x) \quad (x, m, y)(y, n, z) = (x, m + n, z)$$

Topology: generated by the cylinder sets

$$Z(\alpha, \beta) = \{(\alpha y, d(\alpha) - d(\beta), \beta y) \,|\, y \in E^\infty\}$$

• If E has a cycle λ then G_E is not principal: different tails attached to λ^∞ can result in the same path: $\lambda\lambda^\infty = \lambda^2\lambda^\infty$.

• If E has a cycle λ without entry then G_E is not topologically principal: aperiodic paths cannot approximate λ^∞.

A *twist* is an extension of groupoids: an exact sequence

$$\mathbb{T} \times G^{(0)} \xrightarrow{\iota} \Sigma \xrightarrow{q} G \quad \text{such that}$$

- q and ι are continuous groupoid homomorphisms (homeomorphisms of the unit spaces), ι is injective.
- $q^{-1}(G^{(0)}) = \iota(\mathbb{T} \times G^{(0)})$ • $\Sigma/\mathbb{T} \cong G$.

The C^*-algebra $C_r^*(\Sigma; G)$ of the twist is a completion of

$$C_c(\Sigma; G) := \{f \in C_c(\Sigma) \,|\, \forall z \in \mathbb{T} \ \forall \gamma \in \Sigma \ \ f(z \cdot \gamma) = \overline{z} f(\gamma)\}.$$

Renault ('08): Cartan pairs $\mathcal{B} \subseteq \mathcal{A}$ correspond to étale, 2^{nd} countable, locally compact Hausdorff, topologically principal twisted groupoids: $(\mathcal{A}, \mathcal{B}) \cong (C_r^*(\mathcal{G}; \Sigma), C_0(\mathcal{G}^{(0)}))$.

Kumjian ('86): Diagonal pairs...principal groupoids.

Graded twists We say a twist $(\Sigma; G)$ is graded by a discrete abelian group Γ if there are groupoid homomorphisms c_Σ and c_G such that

$$\begin{tikzcd} \mathbb{T} \times G^{(0)} \arrow[r] & \Sigma \arrow[r] \arrow[dr, "c_\Sigma"'] & G \arrow[d, "c_G"] \\ & & \Gamma \end{tikzcd}$$

commutes and $c_G^{-1}(0)$ is topologically principal.

Rmk: such a grading induces a strongly continuous $\hat{\Gamma}$ action on its C^*-algebra characterized by

$$(\omega \cdot f)(\sigma) = \langle \omega \cdot c_\Sigma(\sigma) \rangle f(\sigma) \quad f \in C_c(\Sigma),\, \omega \in \Sigma,$$

providing a Γ-grading on $C_r^*(G; \Sigma)$ with $C_r^*(G; \Sigma)_0 = C_r^*(G; \Sigma)^{\hat{\Gamma}}$.

Theorem 1 (BFPR, '18) If a twist $(\Sigma; G)$ is Γ-graded as above, then $C_0(G^{(0)}) \subseteq C_r^*(G;\Sigma)$ is Γ-Cartan.

Sketch of proof that $C_0(G^{(0)}) \subseteq C_r^*(G;\Sigma)^{\hat{\Gamma}}$ is Cartan:

$$\begin{tikzcd} \mathbb{T} \times G^{(0)} \arrow[r] & \Sigma \arrow[r] \arrow[dr, "c_\Sigma"'] & G \arrow[d, "c_G"] \\ & & \Gamma \end{tikzcd}$$

Let $G_0 = c_G^{-1}(0)$ and $\Sigma_0 = c_\Sigma^{-1}(0)$.

Then $\mathbb{T} \times G^{(0)} \longrightarrow \Sigma_0 \longrightarrow G_0$ is a twist, so by Renault's Theorem $C_0(G^{(0)}) \subseteq C_r^*(\Sigma_0; G_0)$ is Cartan.

Thus it suffices to show that $C_r^*(\Sigma_0; G_0) = C_r^*(G;\Sigma)^{\hat{\Gamma}}$. The inclusion $\subseteq$ is easy; $\supseteq$ requires an approximation argument.

Theorem 2 (BFPR '18): If $(\mathcal{A}, \mathcal{D})$ is a Γ-Cartan pair then there exists a twist

$$\mathbb{T} \times G^{(0)} \longrightarrow \Sigma \longrightarrow G$$

such that $(\mathcal{A}, \mathcal{D}) \cong (C_r^*(\Sigma; G), C_0(G^{(0)}))$.

Proof: rather elaborate. In short, we mimic the Kumjian/Renault construction.

Applying Theorems 1 and 2 in succession we recover the groupoid:

Theorem 3 (BFPR '18) Let $(G; \Sigma)$ be a Γ-graded twist. Then applying Theorem 2 to the Γ-Cartan pair $(C_r^*(\Sigma; G), C_0(G^{(0)}))$ recovers the same twist.

A k*-graph* is a graded countable category $\Lambda = (\Lambda^n, n \in \mathbb{N}^k)$, $\Lambda^m \Lambda^n = \Lambda^{m+n}$, satisfying the *Unique Factorization Property*: If $\lambda \in \Lambda^{m+n}$ then there are unique $\mu \in \Lambda^m,\ \nu \in \Lambda^n$ s.t. $\lambda = \mu\nu$.

The twisted k-graph algebra $C^*(\Lambda, \Phi)$ is defined by Cuntz-Krieger relations twisted by a 2-cocycle $\Phi : \Lambda \times \Lambda \to \mathbb{T}$.

$\mathbb{T}^k$ naturally acts on the algebra via $z \cdot t_\alpha t_\beta^* = z^{d(\alpha)-d(\beta)} t_\alpha t_\beta^*$.
The diagonal $\mathcal{D}$ is Cartan in the fixed-point algebra

$$C^*(\Lambda, \Phi)^{\mathbb{T}^k} \quad = \quad \overline{\operatorname{span}}\{t_\alpha t_\beta^* \,|\, d(\alpha) = d(\beta)\}$$

so we have a twist $(\Sigma; G)$ such that $C^*(\Lambda, \Phi) \cong C_r^*(\Sigma; G)$.

Thus we have recreated the construction of Kumjian-Pask-Sims, who identify a continuous cocycle ξ and a groupoid G_{KPS} with $C^*(\Lambda, \Phi) \cong C^*(G_{KPS}, \xi)$: i.e., $G \cong G_{KPS}$ and $\Sigma \cong \mathbb{T} \times_\xi G_{KPS}$.

Thank you all for attending.

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