Geometry of the set of synchronous quantum correlations

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Correlations

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Figure: Alice

Figure: Bob

n = number of experiments,
m = number of possible outcomes.

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Figure: Bob

$n =$ number of experiments,
$m =$ number of possible outcomes.

A **correlation** is a tuple

$$\{p(i,j|x,y) \geq 0;\quad i,j \in m;\quad x,y \in n\}$$

satisfying

$$\sum_{i,j} p(i,j|x,y) = 1.$$

Quantum correlation sets

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Let A be a C -algebra with a state . Assume

$$\{E_{x,i}\}_{i=1}^{m}, \{F_{y,j}\}_{j=1}^{m} \quad \mathcal{A}$$

where $E_{x,i}F_{y,j} = F_{y,j}E_{x,i}$. Then

$$p(i,j|x,y) = \ (E_{x,i}F_{y,j})$$

defines a **quantum-commuting** correlation.

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Set of all qc correlations: $C_{qc}(n,m)$.

Set of all quantum correlations: $C_q(n,m)$.

Set of all local correlations: $C_{loc}(n,m)$.

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Each $C\ (n; m)$ is convex and satisfies:

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Theorem
(Junge-Navascues-Palazuelos-Perez-Garcia-Scholz-Werner, Fritz, Ozawa)

Connes' embedding conjecture is true if and only if $\overline{C_q(n; m)} = C_{qc}(n; m)$ for every $n; m$.

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Theorem (Paulsen-Severini-Stahlke-Todorov-Winter)

A correlation $p \in C^s_{qc}(n,m)$ iff there exists a C^*-algebra $\mathcal{A}$, $\{E_{x,i}\}_{i=1}^m \subseteq \mathcal{A}$, and a tracial state $\tau : \mathcal{A} \to \mathbb{C}$ such that

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If $\mathcal{A}$ is finite dimensional, $p \in C^s_q(n,m)$. If $\mathcal{A}$ is commutative, $p \in C^s_{loc}(n,m)$.

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What is the geometry of $C^s_q(n; m)$ and $C^s_{qc}(n; m)$?

Theorem (Dykema-Paulsen)

Connes' embedding conjecture is true if and only if $\overline{C^s_q(n; m)} = C^s_{qc}(n; m)$ for every $n; m$.

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We can describe $C_q^s(3;2)$ explicitly as a convex combination of a family of sets in $R^{3^2 2^2}$. In fact,

Theorem (R.)

The set $C_q^s(3;2)$ is closed. Moreover, if p 2 $C_q^s(3;2)$, then there exists a A M$_{16}$*, projection valued measures* $fE_{x;i}g$ A *and a trace such that*

$$p(i;j|x;y) = \ (E_{x;i}E_{y;j}):$$

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When $m = 2$,

$$p(i;jjx;x) = \begin{matrix} r_x & & 0 \\ 0 & 1 & r_x \end{matrix} ;$$

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So the data of $fp(i; jjx; y)g$ is determined by r_x's and $w_{x;y}$'s.

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When $m = 2$,

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So the data of $\{p(i,j|x,y)\}$ is determined by r_x's and $w_{x,y}$'s.

$$p = \begin{pmatrix} r_1 & w_{1,2} & w_{1,3} \\ w_{2,1} & r_2 & w_{2,3} \\ w_{3,1} & w_{3,2} & r_3 \end{pmatrix}, \quad w_{x,y} = w_{y,x}$$

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For each $(r_1, r_2, r_3) \in [0,1]^3$, we will determine the corresponding set of $\{(w_{1,2}, w_{1,3}, w_{2,3})\} \subset \mathbb{R}^3$, denoted $S_{\vec{r}}[C_q^s(3,2)]$.

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Theorem (Alhajjar-R)

$\overline{C^s_{max}(n,m)} = \overline{C^s_q(n,m)}$ for all n,m.

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$$S_d(n_1;n_2;n_3) \quad := \quad f\tfrac{1}{d}(Tr(E_1E_2);Tr(E_1E_3);Tr(E_2E_3)) : Tr(E_x) = n_x g$$

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Goal: Describe $S_d(n_1;n_2;n_3) \quad S_{(n_1=d;n_2=d;n_3=d)}[C^s_q(3;2)]$.

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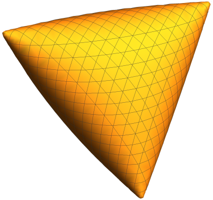

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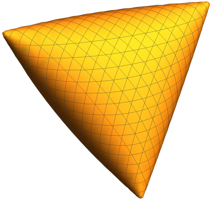

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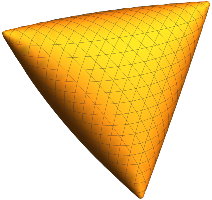

Set $S_d(n) := S_d(n, n, n)$.

Theorem

For every n, $S_{2n}(n) = S_2(1) = S_{(.5,.5,.5)}[C_q^s(3,2)]$ is an affine image of the 3×3 elliptope.

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Lemma

Assume $n_1 + n_2 < d$. Then

$$S_d(n_1; n_2; n_3) \quad \frac{d \quad 1}{d} co\, f S_{d \quad 1}(n_1; n_2; n_3); S_{d \quad 1}(n_1; n_2; n_3 \quad 1) g:$$

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1. Apply lemma many times to find geometry of $S_d(n_1; n_2; n_3)$.

2. Calculate the closure of $[\, S_d(n_1; n_2; n_3)$. This is equal to $\overline{C_q^s(3; 2)}$.

3. Observe that every correlation in $\overline{C_q^s(3; 2)}$ can be realized with A M_{16}.

The main theorem

Theorem (R.)

Assume $r_1 \quad r_2 \quad r_3 \quad 1{=}2,\ \vec{r} = (r_1; r_2; r_3)\ 2\ [0; 1]^3$. *Then*

$$S_{\vec{r}}[C_q^s(3; 2)] = co\, f C_1(\vec{r}); C_2(\vec{r}); C_3(\vec{r}) g$$

where

$$\begin{aligned}
C_1(\vec{r}) &= 2\max(0; r_1 + r_2 + r_3 \quad 1) S_2(1) \\
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Assume $r < 1/2$. Then

$$S_{(r,r,r)}[C_q^s(3,2)] = \mathrm{co}\{\max(0, 6r - 2)S_2(1), 2rS_2(1)\}.$$

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Is $C_q^s(3;2) = C_{qc}^s(3;2)$?

Thanks for your attention!