Localization near the edge for the Anderson Bernoulli model on the two-dimensional lattice

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Joint work with Charles Smart (University of Chicago)

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- $(Vu)(x) = V_x u(x)$ is a random potential;
- $V_x \in \{0, 1\}$ are i.i.d. Bernoulli variables;
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Remark: For concreteness we assume $\delta = 1$, and $\mathbb{P}(V_x = 0) = 1/2$.

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holds for any ψ satisfying the following:

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Remark: the above is usually referred to as spectral localization. There is also a notion of dynamic localization which is more directly related to the transport of the electron.

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Remark: Except for a spectral measure 0, each spectrum value has a polynomially bounded solution to the eigenfunction equation.

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- If the lattice is replaced by the continuum $\mathbb{R}^d$, then H almost surely has Anderson localization in $[0, \epsilon]$ (Bourgain–Kenig 05).

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Theorem. (Exponential decay for resolvent; D.–Smart 18)
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Remark: Resolvent decay was established for $\mathbb{R}^d$ in Bourgain–Kenig 05, via a powerful framework of multi-scale analysis.

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Unique continuation principle (UCP) on $\mathbb{R}^d$: if $u \in C^2(B_R)$, $|u(0)| = 1$, $|\Delta u| \leq \alpha |u|$, and $|u| \leq \alpha$, then for some $\beta > 0$

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UCP is a key ingredient in Bourgain–Kenig 05 for $\mathbb{R}^d$ which does not hold for $\mathbb{Z}^d$, even for harmonic functions.

- In $\mathbb{Z}^2$ there exists a non-zero harmonic function which vanishes on half of the plane.
- In $\mathbb{Z}^3$ there exists a non-zero harmonic function which vanishes except on a plane.

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The key point is that every site responds to the potential perturbation by UCP.

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Thus, we only need a version of UCP on $\mathbb{Z}^d$ with size of support $\gg \sqrt{volume}$.

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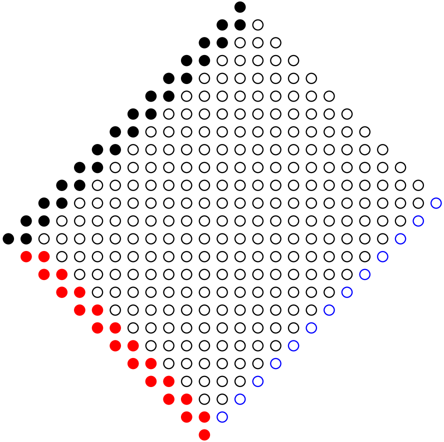
Key challenge for us is to deal with potentials.

• In the worst case potential, there exists a harmonic function supported only on a diagonal. We have to use “randomness” of the potential in some way.

• A key step in Buhovsky-Logunov-Malinnikova-Sodin is to study the propagation of the harmonic function with 0-boundary on west diagonals and input on the south diagonals.

A key step in Buhovsky-Logunov-Malinnikova-Sodin



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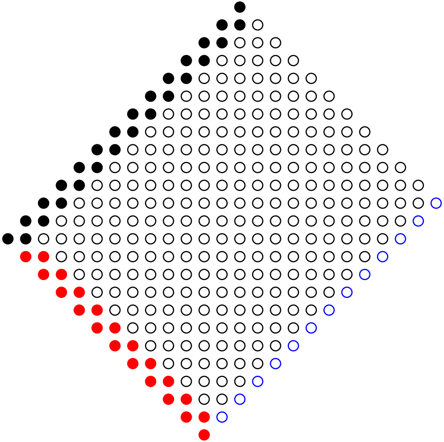

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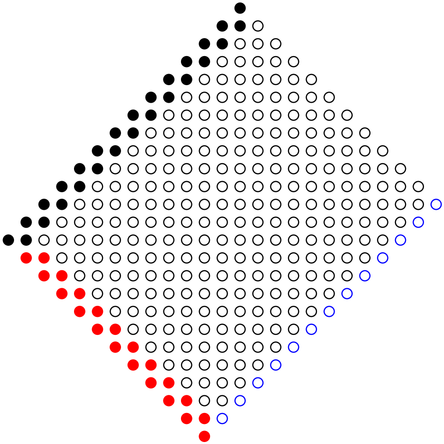

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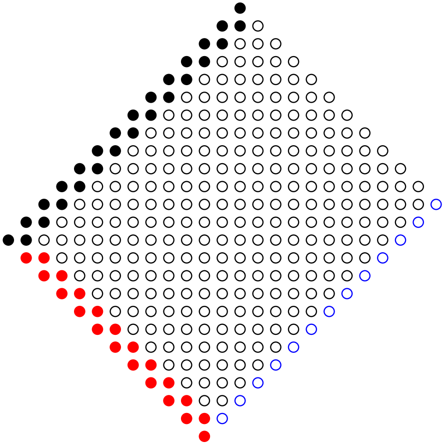

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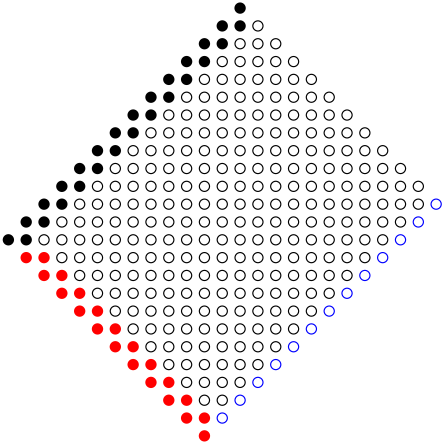

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Conclusion: If blue circles are bounded on half fraction, it is bounded on all (up to an exponential factor).

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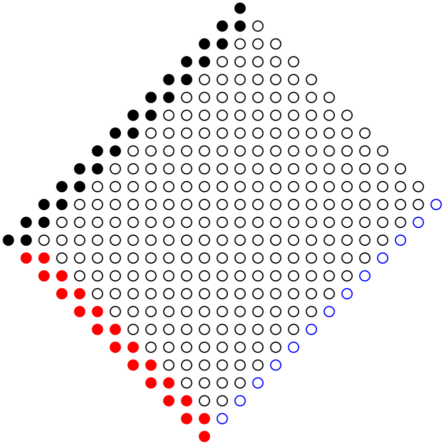

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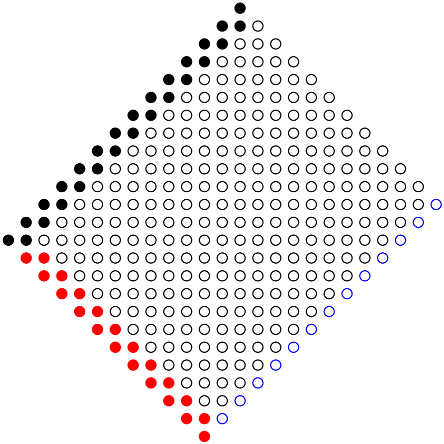

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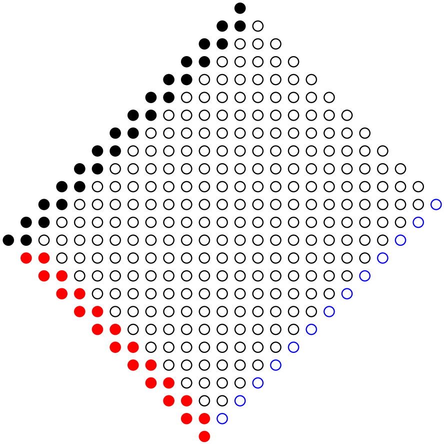

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Our main idea: Show that if the max on red bullets is 1, then at least a linear fraction of blue circles is lower bounded by exponential decay.

- Apply union bound with regularity on red input.
- Regularity is poor due to inhomogeneity for influences from different red bullets.
- Thus can only work in a thin rectangle.

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• But, in $d = 3$, it seems even with worst potential the support of any solution is at least two-dimensional.

• Li–Zhang proved a weaker version: for $d = 3$, with any potential any solution has support with exponential lower bound on at least $N^{3/2+\epsilon}$ vertices.

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Happy birthday to HT!