Long term transients and noise induced oscillations in a predator-prey model with timescale separation.

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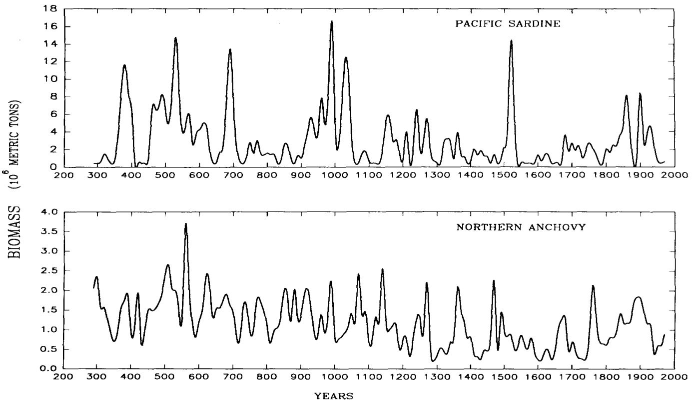
New Mathematical Methods for Complex Systems in Ecology, BIRS, Canada, July 28-Aug 02, 2019.

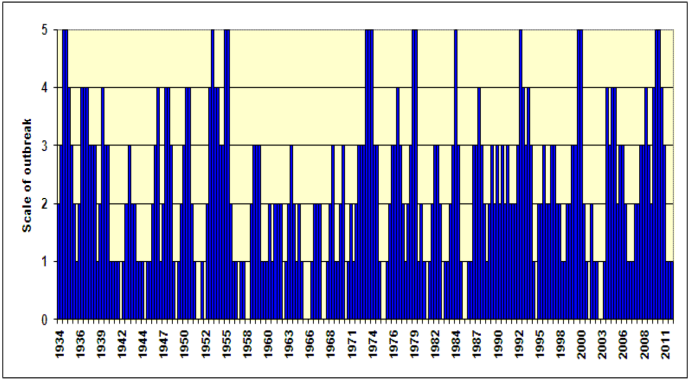
Booms and bust are common in Nature



1700-year series of Pacific sardine and northern anchovy biomasses off California and Baja California. *Source: “History of Pacific Sardine and Northern Anchovy Populations”, Baumgartner et. al. CALCOFI Rep. 33 (1992).*

Locust outbreaks

Outbreak intensities of the ***Australian plague Locusts***



http://www.agriculture.gov.au/pests-diseases-weeds/locusts

Outline

- A three species predator-prey model with timescale separation:
  1. Long transient dynamics in a regime close to a supercritical Hopf bifurcation.
  2. Underlying mechanism leading to transition to asymptotics.
  3. Predict the occurrence of an outbreak based on the analysis.
  4. Effect of stochasticity near the Hopf bifurcation.
  5. Distribution of the return time between two outbreaks.

Nondimensional Model

x - prey density,
y, z - predators' densities.

$$\begin{cases} x' &= x\left(1 - x - \frac{y}{\beta_1 + x} - \frac{z}{\beta_2 + x}\right) := x\phi(x,y,z) \\ y' &= \varepsilon y\left(\frac{x}{\beta_1 + x} - c - \alpha_{12} z\right) := \varepsilon y \chi(x,y,z) \\ z' &= \varepsilon z\left(\frac{x}{\beta_2 + x} - d - \alpha_{21} y - hz\right) := \varepsilon z \psi(x,y,z). \end{cases} \tag{1}$$

Assumptions on the Parameters

- $0 < \varepsilon << 1$.
- $0 < \beta_1,\ \beta_2 < 1$, where β_1, β_2 are the predation efficiencies.
- We will assume that $0 < c,\ d,\ \alpha_{12},\ \alpha_{21} < 1$.

Preliminaries

fast-time t $\begin{cases} x' &= x\phi(x,y,z) \\ y' &= \varepsilon y\chi(x,y,z) \\ z' &= \varepsilon z\psi(x,y,z), \end{cases}$ slow-time s $\begin{cases} \varepsilon\dot{x} &= x\phi(x,y,z) \\ \dot{y} &= y\chi(x,y,z) \\ \dot{z} &= z\psi(x,y,z) \end{cases}$

where $s = \varepsilon t$. Here x is the fast variable and y, z are the slow variables.

As $\varepsilon \to 0$, we obtain reduced and layer problems respectively:

Reduced problem

$$\begin{cases} 0 &= x\phi(x,y,z) \\ \dot{y} &= y\chi(x,y,z) \\ \dot{z} &= z\psi(x,y,z) \end{cases}$$

Layer problem

$$\begin{cases} x' &= x\phi(x,y,z) \\ y' &= 0 \\ z' &= 0. \end{cases}$$

Critical Manifold: Equilibria of the layer problem

$$\mathcal{M} = T \cup S = \{(x, y, z) : x = 0\} \cup \{(x, y, z) : \phi(x, y, z) = 0\}.$$

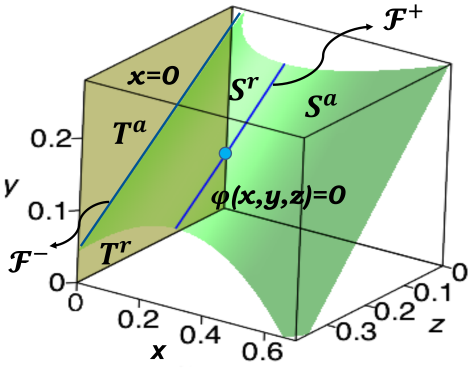
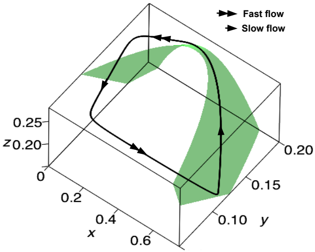
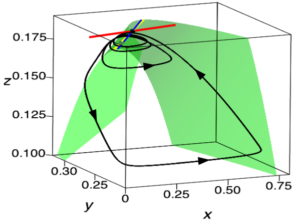


Figure: The critical manifold $\mathcal{M}$.

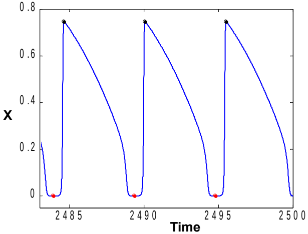
Oscillatory dynamics exhibited by the full-system

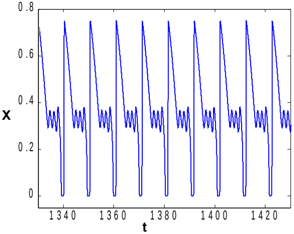


(a) Relaxation oscillations

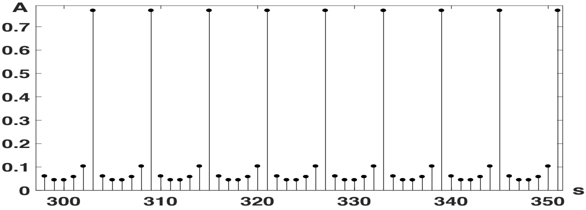


(b) Mixed-mode oscillations

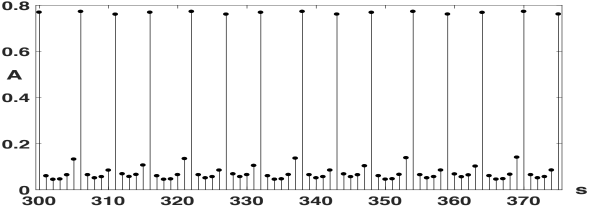




Variation in the amplitudes of SAOs and LAOs in an MMO orbit



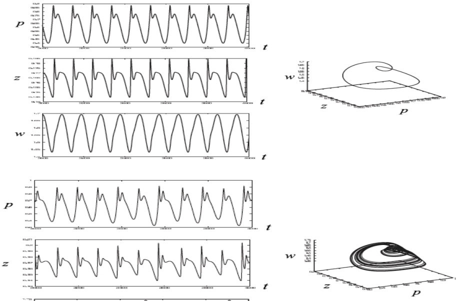
Relative amplitudes of SAOs and LAOs of a 1^5 MMO orbit



Relative amplitudes of SAOs and LAOs of a chaotic MMO orbit

MMOs in Ecology

- MMOs have been observed in the study of chaotic dynamics in three species food-prey-predator model (an extension of Hastings-Powell model) (Peet et. al. JTB (2005)).
- Also observed in population dynamics of phytoplankton-zooplankton freshwater ecosystem with dormancy (Kuwamura & Chiba, Chaos (2008)).

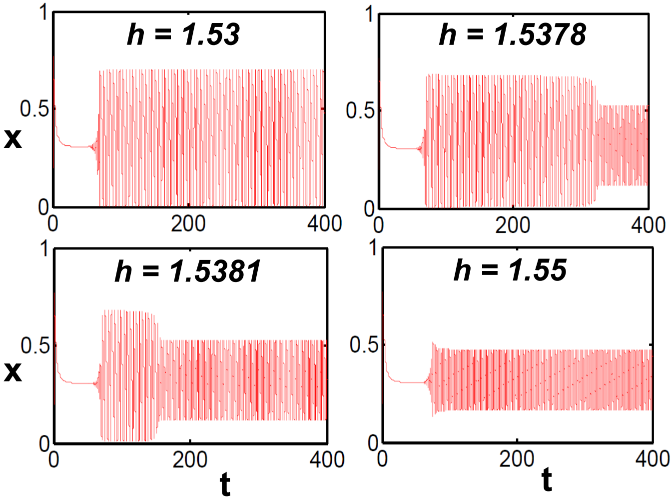


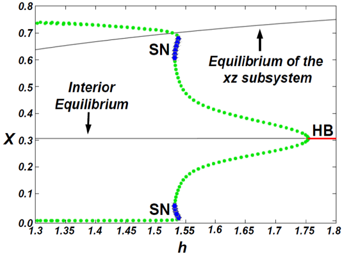
Local mechanisms generating SAOs in MMOs

Desroches et. al. SIAM Review (2012)

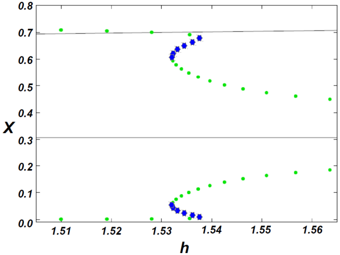
- Passage near special points such as a folded node singularity/canard point (Szmolyan & Wechselberger JDE (2001))
- Interaction between the slow manifold and the unstable manifold of the equilibrium (Guckenheimer SIADS (2008)).
- Generalized canard mechanism (occurs near FSN II bifurcation) (Krupa & Wechselberger JDE (2010)).

Transient dynamics in the full-system





(e) A 1-parameter bifurcation diagram in *h*

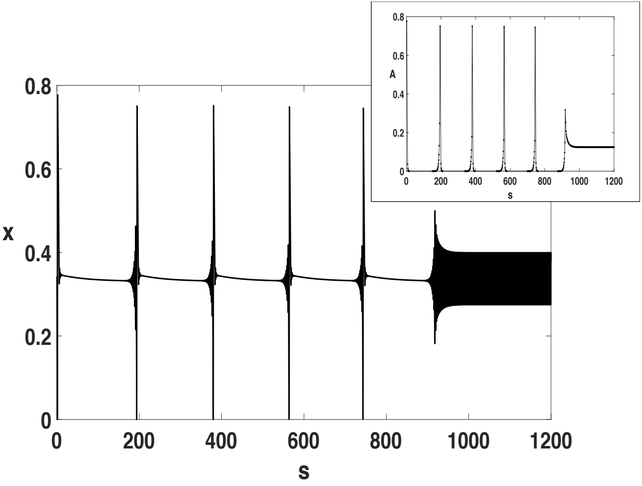


(f) A zoomed view showing LPC

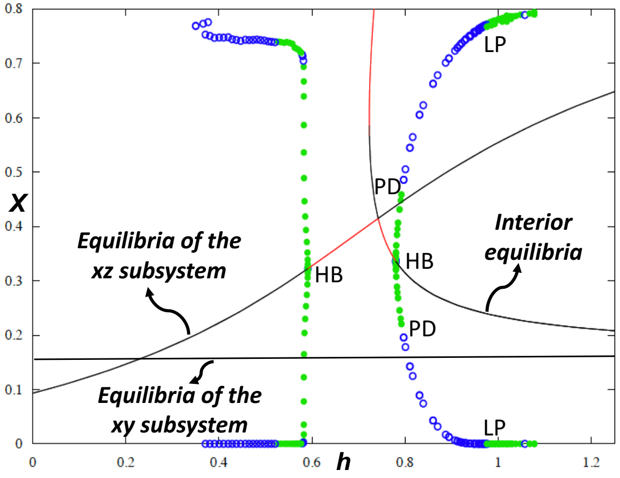
Figure: S. and Chakraborty Thakur, Ecol. Compl. (2017).

Similar transient dynamics were observed in a stage structured population model with time delay and Allee effect (Morozov et.al. JTB (2016)).

Long term transients with large fluctuations!



One-parameter bifurcation diagram in h



Dynamics past the HB

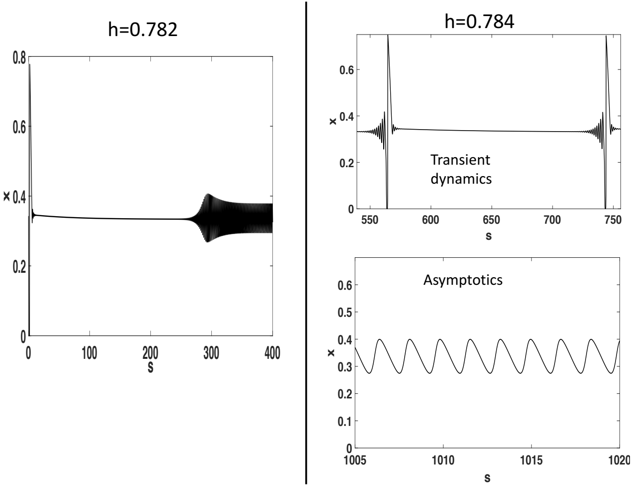
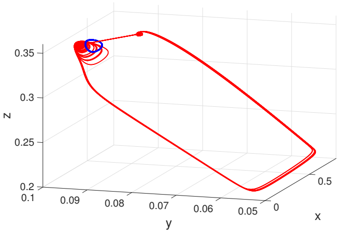
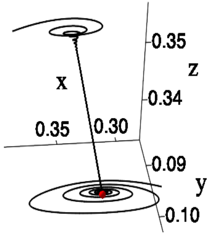


Figure: Transient dynamics in the Hopf regime.

Ghost attractor near the Hopf regime



(a) Chaotic MMO orbit and the SAO attractor



(b) Local dynamics near the equilibrium

Similar local dynamics with different fates

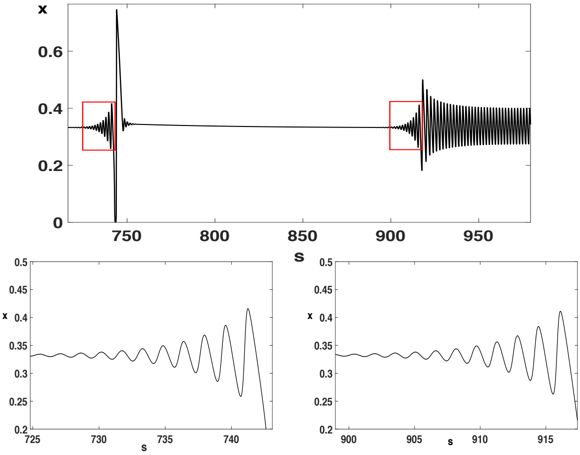


Figure: Dynamics preceding an LAO (left). Dynamics preceding an SAO attractor (right).

Normal form near FSN II point

Theorem

System (1) can be written in the normal form as

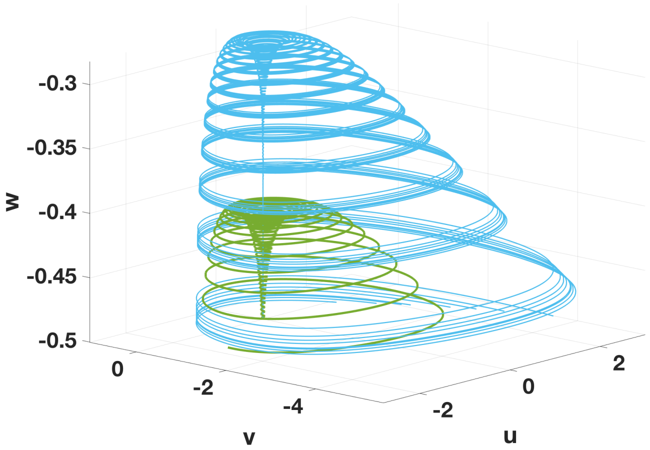
$$\begin{cases} \frac{du}{d\tau} &= v + \frac{u^2}{2} + \delta(\alpha(h)u + F_{13}uw + \frac{1}{6}F_{111}u^3) + O(\delta^2) \\ \frac{dv}{d\tau} &= -u + O(\delta^2) \\ \frac{dw}{d\tau} &= \delta(H_3 w + \frac{1}{2}H_{11}u^2) + O(\delta^2) \end{cases} \quad (2)$$

with $\delta = O(\sqrt{\varepsilon})$ *and* $\tau = s/\delta$, *where*

$$\alpha(h) = A_0(\bar{x}, \bar{y}, \bar{z}, \bar{h})\frac{(h - \bar{h})}{\varepsilon} - B_0(\bar{x}, \bar{y}, \bar{z}, \bar{h})$$

and $(\bar{x}, \bar{y}, \bar{z}, \bar{h})$ *is a non-degenerate singular Hopf point. The constants* $F_{13}, F_{111}, H_3, H_{11}, A_0, B_0$ *can be explicitly computed in terms of derivatives of* u, v *and* w *at* $(\bar{x}, \bar{y}, \bar{z}, \bar{h})$.

Dynamics of the system in normal form



Dynamics in the singular limit

For $\delta = 0$:

$$\begin{cases} \frac{du}{d\tau} &= v + \frac{u^2}{2} \\ \frac{dv}{d\tau} &= -u \\ \frac{dw}{d\tau} &= 0. \end{cases} \tag{3}$$

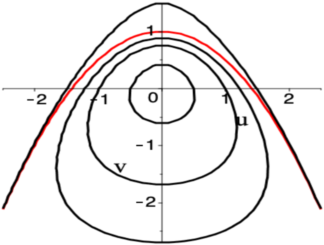


Figure: Orbits outside the red curve become unbounded.

Red curve: $(u^2 + 2v - 2) = 0$.

Existence of a quasi-separatrix in the *uv*- subsystem

For $\delta > 0$, replace $w(\tau)$ by a parameter λ.

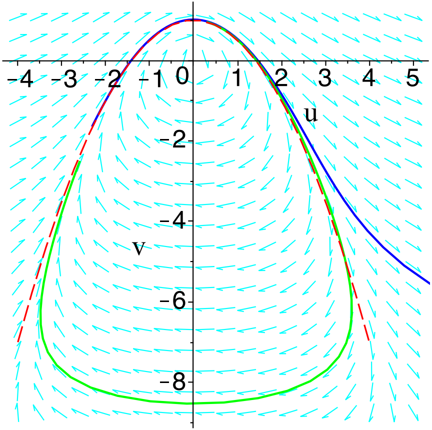


Figure: Behavior of two trajectories starting on the parabola $(u^2 + 2v - 2) = 0$ for a fixed λ.

Dynamic passage through a canard explosion

A 2-parameter bifurcation diagram of the *uv* subsystem

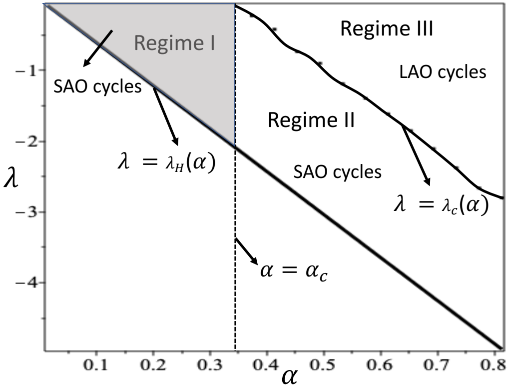
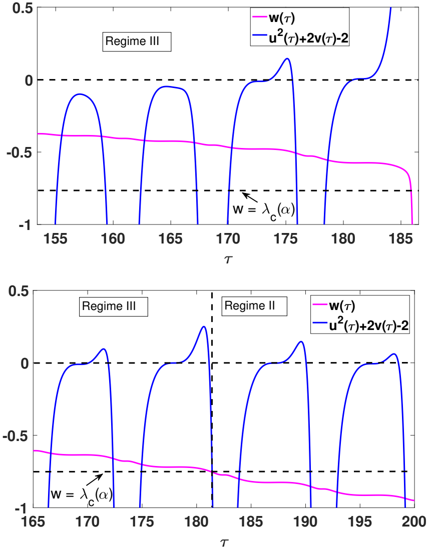


Figure: Supercritical Hopf: $\lambda = \lambda_H(\alpha)$. A canard explosion occurs at $\lambda = \lambda_c(\alpha)$.

Returning to the full normal form



Effect of stochasticity

S. Chaos 2017.

$$\begin{cases} dX = \frac{X}{\varepsilon}\left[1 - X - \frac{XZ}{\beta_2+X}\right] ds + \frac{\sigma_1}{\sqrt{\varepsilon}} X dW_1(s) \\ dZ = Z\left[\frac{X}{\beta_2+X} - d - hZ\right] ds + \sigma_2 Z dW_2(s). \end{cases} \quad (5)$$

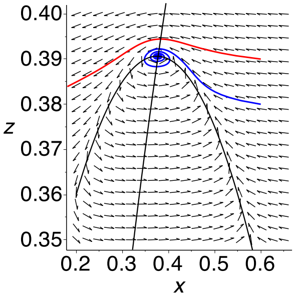
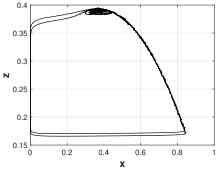
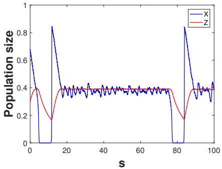


Figure: Locally asympotically stable equilibrium.

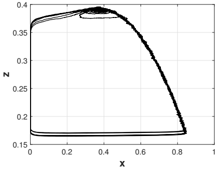
Noise-induced mixed-mode oscillations



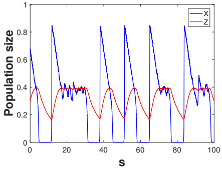
(a) $\sigma_1 = \sigma_2 = 0.003$



(b) Corresponding time series



(c) $(\sigma_1, \sigma_2) = (0.005, 0.004)$

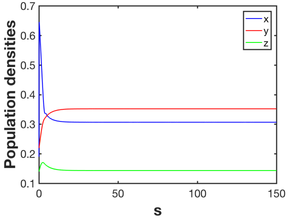


(d) Corresponding time series

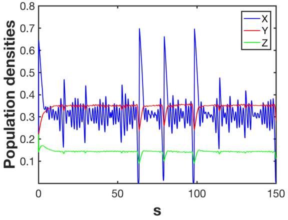
Figure: Random sample paths for different noise intensities.

Demographic stochasticity on the 3d model

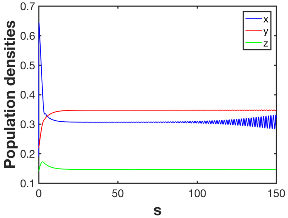
S. and C. Kuehn, Chaos (2018)

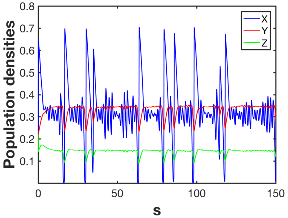


(a) Deterministic time series



(b) Isolated spikes





Random population fluctuations computed for two sample paths

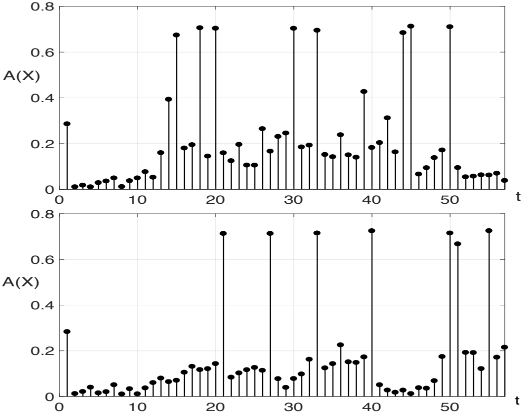
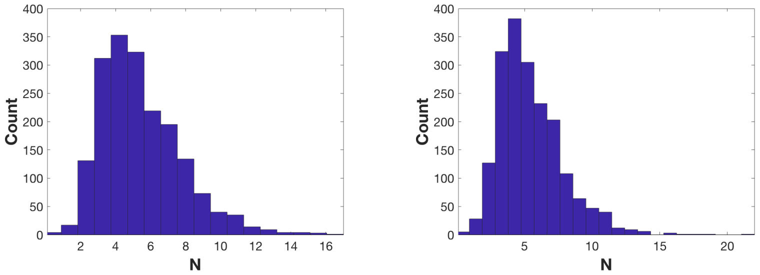


Figure: Amplitude of oscillations in the prey population.

Distribution of the random number of small oscillations between two large oscillations with varying noise

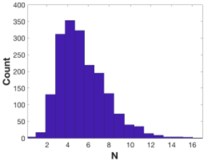


(a) $\sigma_1 = 10^{-6}, \sigma_2 = 10^{-3}, \sigma_3 = 10^{-5}$.

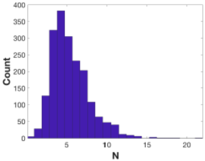
(b) $\sigma_1 = 10^{-6}, \sigma_2 = 10^{-3}, \sigma_3 = 10^{-4}$

Figure: S. and C. Kuehn, Chaos (2018).

Similarity with the distribution of the return time of larch budmoth outbreaks.



(a) $\sigma_1 = 10^{-6}, \sigma_2 = 10^{-3}, \sigma_3 = 10^{-5}$.



(b) $\sigma_1 = 10^{-6}, \sigma_2 = 10^{-3}, \sigma_3 = 10^{-4}$

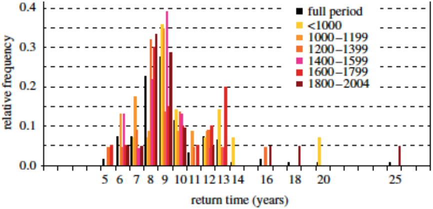


Figure: Source: “1200 years of regular outbreaks in alpine insects", Esper et. al. Proc. Royal Soc. B (2007).

Summary

- Long time transient dynamics in the form of MMOs are observed in the deterministic model near the supercritical Hopf bifurcation.
- These dynamics occur as a result of interplay between SAOs and a suitable global return mechanism.
- The normal form is used to study the mechanism that organizes the SAOs in an MMO cycle - a dynamic passage through a canard explosion.
- The normal form helps to predict the possibility of an another large cycle.
- Noise-induced MMOs provide a realistic representation of dynamics occurring between outbreaks.

Thank you for listening!