Localization for a disordered XXZ spin chain

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BIRS workshop, August 2019

Many body systems - Setup

Let $\mathscr{V}$ be a finite set of vertices. We consider quantum systems with a Hilbert space $\mathscr{H}_x$ at each vertex $x \in \mathscr{V}$. For $\mathscr{X} \subset \mathscr{V}$, the Hilbert space associated with $\mathscr{X}$ is the tensor product $\mathscr{H}_{\mathscr{X}} = \otimes_{x \in \mathscr{X}} \mathscr{H}_x$. The algebra of observables in X is denoted by $\mathscr{A}_{\mathscr{X}} := B(\mathscr{X})$. It naturally extends to $\mathscr{A}_{\mathscr{V}}$ for any $\mathscr{X}$ by identifying $A_{\mathscr{X}} \in \mathscr{A}_{\mathscr{X}}$ with

$$A_{\mathscr{X}} \otimes I_{\mathscr{X}^c} \in \mathscr{A}_{\mathscr{V}}.$$

An interaction for such a system is a map $\Phi : \mathscr{V} \to \mathscr{A}_{\mathscr{V}}$ such that $\Phi(\mathscr{X}) \in \mathscr{A}_{\mathscr{X}}$ and $\Phi^*(\mathscr{X}) = \Phi(\mathscr{X})$. The Hamiltonian H on $\mathscr{H}_{\mathscr{V}}$ is defined by

$$H = \sum_{\mathscr{X} \subset \mathscr{V}} \Phi(\mathscr{X}).$$

The dynamics of the model is the one-parameter group of automorphisms, $\{\tau_t\}_{t \in \mathbb{R}}$, defined by $\tau_t(A) = \exp(itH) A \exp(-itH)$, $A \in \mathscr{A}_{\mathscr{V}}$.

Many body systems - Spin systems

- Spin systems: Each $\mathscr{H}_x$ is $\mathbb{C}^2$. The algebra of diagonal observables is generated by the identity and Pauli matrices,

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

  The interaction is typically of the nearest neighbor type: We think of $\mathscr{V}$ as the graph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ and $\Phi(\mathscr{X}) \neq 0$ only if the graph diameter of $\mathscr{X} \leq 1$. The singlet interaction is called a magnetic potential.

Lieb–Robinson bound, information propagation

For spin systems (with short range interaction, bounded interactions) one has

1. Lieb–Robinson bound: There exist constants $C, m > 0$ and $v \geq 0$ such that $\|[A_{\mathscr{S}}, \tau_t(B_{\mathscr{T}})]\| \leq C \|A_{\mathscr{S}}\| \, \|B_{\mathscr{S}}\| \, e^{-m(\ell - v|t|)}$, where $\ell = \mathrm{dist}(\mathscr{S}, \mathscr{T})$, the graph distance between $\mathscr{S}, \mathscr{T} \subset \mathscr{V}$.

2. Information propagation, locality (Bravyi–Hastings–Verstraete): There exist constants $C, m > 0$ and $v \geq 0$ such that

$$\left\| \tau_t(A_{\mathscr{S}}) - \mathrm{tr}_{(\mathscr{S}+[-\ell,\ell])^c} \, \tau_t(A_{\mathscr{S}}) \right\| \leq C \, \|A_{\mathscr{S}}\| \, e^{-m(\ell - v|t|)}, \quad \ell \in \mathbb{N}.$$

   Note that $\mathrm{tr}_{(\mathscr{S}+[-\ell,\ell])^c} \, \tau_t(A_{\mathscr{S}}) \in \mathscr{A}_{\mathscr{S}+[-\ell,\ell]}$.

   - Equivalent to Lieb–Robinson bound.

Exponential clustering for gapped ground states

Are there more general statements valid for analytic functions of the Hamiltonian? The difficulty: Even though H is local, its powers (and hence analytic functions) are not! It does not happen in the one particle picture. But:

Theorem (Exponential clustering theorem, Hastings)

Assume that the ground state E_o of a local H is separated from the rest of $\sigma(H)$ by a finite gap g. Then there exist $C, m > 0$ (that depend on g) such that for any normalized $\psi \in N_{H-E_o}$ we have, with $\ell = \mathrm{dist}(\mathscr{S}, \mathscr{T})$,

$$|\langle \psi, A_{\mathscr{S}} B_{\mathscr{T}} \psi \rangle - \langle \psi, A_{\mathscr{S}} \psi \rangle \langle \psi, B_{\mathscr{T}} \psi \rangle| \leq C \|A_{\mathscr{S}}\| \|B_{\mathscr{T}}\| e^{-m\ell}.$$

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- If E_o is simple, the statement can be reformulated in terms of the spectral projection P_{E_o}, which is a smooth function of H in the gapped case.
- Using the combinatorial approach of Arad–Kitaev–Landau–Vazirani, the same statement can be shown to hold for other gapped simple evals of H.

Area law for the gapped systems

- Let $\mathscr{G} = \mathbb{Z}^d$, $\Lambda_L = [-L, L]^d \cap \mathbb{Z}^d$. For a local, gapped H with $\psi \in N_{H-E_o}$, we consider $\rho = |\psi\rangle\langle\psi|$, and its *entanglement entropy*

$$S_{L;\ell} := -\operatorname{tr}_{\Lambda_\ell} \rho^{\Lambda_\ell} \log \rho^{\Lambda_\ell}, \quad \ell \ll L.$$

- How $S_{L;\ell}$ behaves in the thermodynamic limit $L \to \infty$? Hastings showed that it satisfies *area law* in $d = 1$ case, i.e., $S_{L;\ell} \leq C = C\ell^{d-1}$.
- Brandao–Horodecki: In $d = 1$ the exponential clustering for a state ψ (not necessary a ground state) implies area law. However, it is in general false in higher dimension (data hiding states).
- Clearly shows that gapped ground states for local Hamiltonians do not thermalize (otherwise $S_{L;\ell} \sim \ell^d$)!

XYZ Heisenberg model with a uniform magnetic potential

$$H_h = \sum_{n=-L}^{L-1} \left(J_x \sigma_n^x \sigma_{n+1}^x + J_y \sigma_n^y \sigma_{n+1}^y + J_z \sigma_n^z \sigma_{n+1}^z \right) + h \sum_{n=-L}^{L} \sigma_n^z.$$

It is common to name the model depending on the values of $J_{x,y,z}$:

- XYZ model: J_x, J_y, J_z are distinct;
- XXZ model: $J_x = J_y \neq J_z$;
- XXX model: $J_x = J_y = J_z$;
- XY (or XX) model: $J_x = J_y$, $J_z = 0$;
- Ising model: $J_y = J_z = 0$.

The XYZ model is a protatypical model in statistical physics, one-dimensional magnetism and quantum communication.

Solvability of XYZ model

- XXX chain is exactly solvable (Baxter, Takhtadzhan–Faddeev using algebraic Bethe Ansatz, etc.); satisfies the Yang–Baxter equation.
- XXZ chain is diagonalizable by the algebraic Bethe Ansatz as well (Babbitt–Thomas–Gutkin, Faddeev, Borodin, Corwin, etc). The analytic Bethe Ansatz also reveals the structure of the spectrum near the ground state (more on it later).
- XY chain can be mapped to a free Fermion system via the Jordan–Wigner transform and solved quite explicitly (quasi-free systems).
- The good model to test MBL: replace the uniform magnetic field with the random one,

$$H_\omega = \sum_{n=-L}^{L-1} \left(J_x \sigma_n^x \sigma_{n+1}^x + J_y \sigma_n^y \sigma_{n+1}^y + J_z \sigma_n^z \sigma_{n+1}^z \right) + \sum_{n=-L}^{L} \omega_n \sigma_n^z.$$

Random XY chain

- The analysis reduces effectively to the one of 1d (block) random Schrödinger operator (plus the control of the J-W transformation), Stolz and collaborators (Abdul-Rahman; Hamza; Sims).
- The results include almost sure p.p. spectrum, the zero velocity Lieb–Robinson bound, exponential clustering of eigenvectors (Sims–Warzel), and area law in in the strong disorder regime.

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- Unfortunately, does not help to develop intuition about the disordered spin chains... Rather, it goes other way around - whatever results one can get without relying on exact solvability can be tested on the random XY spin chain! The understanding (in the XXZ context, E.–Klein–Stolz) of the results for energy intervals led to the study of their counterparts in the XY model...

Random XXZ chain

- The next problem (in the level of difficulty) is a random XXZ chain.
- Not exactly solvable, but in addition to energy has another conserved quantity - the particle number operator (magnitization) $\mathscr{N}$:

$$[H,\mathscr{N}]=0, \quad \mathscr{N}=\sum_{i=-L}^{L}\mathscr{N}_i, \quad \mathscr{N}_i=\tfrac{1}{2}\left(1-\sigma_i^z\right).$$

  Note that $\mathscr{N}_i$ is the projection onto the down-spin state on site i.
- Convenient to work in a (computational) basis $\beta := \otimes_{i=-L}^{L} e_i^{\alpha}$, $\alpha\in\{0,1\}$, $e_i^0=\uparrow$, $e_i^1=\downarrow$.
- Scaling and shifting energy, we may consider

$$H_\omega=\tfrac{1}{4}\sum_{i=-L}^{L-1}\left(1-\sigma_i^z\sigma_{i+1}^z\right)-\tfrac{1}{\Delta}\left(\sigma_i^x\sigma_{i+1}^x+\sigma_i^y\sigma_{i+1}^y\right)+\lambda\sum_{i=-L}^{L}\omega_i\mathscr{N}_i.$$

Random XXZ chain: 3 parts

$H_\omega \geq 0$ as can be seen from $H_\omega = -\frac{1}{2\Delta}D + \left(1 - \frac{1}{\Delta}\right)W + \lambda V_\omega$, where:

- $D = \sum_{i=-L}^{L-1} d_{i,i+1}$ with $2d_{i,i+1} = \sigma_i^z\sigma_{i+1}^z + \sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y - 1$. This is the Laplacian in disguise:
  $d_{i,i+1}\left(\uparrow_i \otimes \uparrow_{i+1}\right) = d_{i,i+1}\left(\downarrow_i \otimes \downarrow_{i+1}\right) = 0,$
  $d_{i,i+1}\left(\uparrow_i \otimes \downarrow_{i+1}\right) = \downarrow_i \otimes \uparrow_{i+1} - \uparrow_i \otimes \downarrow_{i+1},$
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  $d_{i,i+1}\left(\uparrow_i \otimes \uparrow_{i+1}\right) = d_{i,i+1}\left(\downarrow_i \otimes \downarrow_{i+1}\right) = 0,$
  $d_{i,i+1}\left(\uparrow_i \otimes \downarrow_{i+1}\right) = \downarrow_i \otimes \uparrow_{i+1} - \uparrow_i \otimes \downarrow_{i+1},$
  $d_{i,i+1}\left(\downarrow_i \otimes \uparrow_{i+1}\right) = \uparrow_i \otimes \downarrow_{i+1} - \downarrow_i \otimes \uparrow_{i+1}.$
- Think about $\downarrow$ as 'particle', $\uparrow$ as 'background', then $\uparrow_i \otimes \downarrow_{i+1} \to \downarrow_i \otimes \uparrow_{i+1}$ means that a particle hopped to the left;
- $$W = \frac{1}{4}\sum_{i=-L}^{L-1}\left(1 - \sigma_i^z \sigma_{i+1}^z\right) = \frac{1}{8}\sum_{i=-L}^{L-1}\left(\sigma_i^z - \sigma_{i+1}^z\right)^2 = \frac{1}{2}\sum_{i=-L}^{L-1}\left(\mathcal{N}_i - \mathcal{N}_{i+1}\right)^2.$$

  This is the interacting potential term (counts number of particles' clusters);
- $V_\omega = \sum_{i=-L}^{L} \omega_i \mathcal{N}_i$ is the random potential.

Random XXZ chain: Structure of spectrum

- Illustration for D: A single particles' cluster

$$D_{off}\,(\ldots\uparrow\uparrow\uparrow\uparrow\downarrow\downarrow\downarrow\uparrow\uparrow\uparrow\uparrow\ldots) = \ldots\uparrow\uparrow\uparrow\downarrow\uparrow\downarrow\downarrow\uparrow\uparrow\uparrow\uparrow\ldots + \ldots\uparrow\uparrow\uparrow\uparrow\downarrow\downarrow\uparrow\downarrow\uparrow\uparrow\uparrow\ldots.$$

- The number of particles is conserved, but clusters can split. So associating a 'quasi-particle' with each cluster is not such a great idea.
- We will consider the Ising phase of XXZ ($\Delta > 1$).
- The ground state Ω: all spins are up (zero particles state, vacuum). $H_\omega \Omega = 0$. Requires some natural boundary conditions.
- Energy goes up with (a) the number of clusters (thanks to W) and (b) the number of particles N (thanks to V_ω). A 'typical' N-particles state has $\sim N$ clusters, so its energy is $\sim N$ as well.
- Morale: Typical states in XXZ model correspond to the temperature $T \sim N$: ($E = \operatorname{tr} H\rho(T)$). In the limit $N \to \infty$, $T \to \infty$. The hard problem...

Locality in XXZ chain

- Strategy: Find first some property that holds for the analytic functions of the Hamiltonian, and then verify it for eigenprojections. Works in the one particle localization!
- Given $S \subset [-L, L] \cap \mathbb{Z}$, $S \neq \emptyset$, we define projections $P_\pm(S)$ by

$$P_+^{(S)} = \otimes_{j\in S}(1 - \mathcal{N}_j) \text{ and } P_-^{(S)} = 1 - P_+^{(S)}.$$

- In particular, $P_-^{(S)} P_+^{(T)} = P_+^{(T)} P_-^{(S)} = 0.$ whenever $S \subset T$.

Theorem (Locality)

Let $S \subset T$ be intervals, and let $\ell = \operatorname{dist}(S, T^c) - 1 \geq 1$. Then we have

$$\left\| P_-^{(S)} f(H) P_+^{(T)} \right\| \leq C \|\hat{f}\|_\infty \ell^2 e^{-\frac{1}{2}\ell} + \int_{v|t|>\ell} \left|\hat{f}(t)\right| dt. \qquad (1)$$

- *Useful if $\hat{f}(t)$ decays rapidly with t.*

Random XXZ chain: Results for the droplet spectrum

Fix $\delta \in (0,1)$. We set $I_k = \left[(k-\delta)\left(1-\frac{1}{\Delta}\right), (k+1-\delta)\left(1-\frac{1}{\Delta}\right)\right]$.

- The spectrum $\sigma(H)$ is contained in $\{0\} \cup \left[1-\frac{1}{\Delta}, \infty\right)$.
- The spectrum of H_ω in the interval $\left[1-\frac{1}{\Delta}, 2\left(1-\frac{1}{\Delta}\right)\right]$ is called the **droplet spectrum**.

Theorem (Droplet localization (E.–Klein–Stolz '17))

There exists a constant $K > 0$ with the following property: If

$$\lambda\sqrt{\Delta-1}\min(1,\Delta-1) \geq K,$$

then there exist constants $C < \infty$ and $m > 0$ such that

$$\mathbb{E}\left(\sum_{E\in\sigma(H_\omega)\cap I_1} \|\mathcal{N}_i\psi_E\|\,\|\mathcal{N}_j\psi_E\|\right) \leq Ce^{-m|i-j|} \text{ for all } i,j \in [-L,L], \qquad (2)$$

uniformly in L.

Remarks

- The 'finite temperature' regime ($E = \operatorname{tr} \rho(T) H$).
- Roughly speaking, (2) indicates that the droplet eigenstates of H resemble a single cluster (droplet) of down-spins in a sea of up-spins.
- SULE: With large probability, for each $E \in \sigma(H_\omega) \cap I_1$, we can find $x_E \in \mathbb{N}$ such that for $i \in \mathbb{N}$,

$$\|\mathcal{N}_i \psi_E\| \leq C_L e^{-m|x_E - i|}. \tag{3}$$

- Beaud–Warzel '17: Results on N particle sectors of XXZ model, (2) can be deduced from them.
- The result cannot be extended to energies above the droplet regime (due to the presence of many cluster states).
- Let $S_\ell(i) = [i - \ell, e + \ell] \cap \mathbb{Z}$, then the following is equivalent to (3):

$$\left\| P_+^{S_\ell(x)} \psi_E \right\| \leq C_L e^{-m\ell}. \tag{4}$$

Consequences of Eq. 2:Exponential clustering

Notation

- (a) Time evolution $\tau_t^I(X)$ of an observable X: $\tau_t^I(X) = e^{itH_I} X e^{-itH_I}$;
- (b) Correlator $R_{X,Y}(\psi)$ of a (normalized) state ψ:
  $R_{X,Y}(\psi) = |\langle\psi, XY\psi\rangle - \langle\psi, X\psi\rangle\langle\psi, Y\psi\rangle|$.

Theorem (Dynamical exponential clustering)

Eq. 2 implies that for all all local observables X and Y with $\max\operatorname{supp} X < \min\operatorname{supp} Y$,

$$\mathbb{E}\left(\sup_{t\in\mathbb{R}} \sum_{E\in\sigma(H)\cap I_1} R_{\tau_t^{I_1}(X),Y}(\psi_E)\right) \le C\,\|X\|\,\|Y\|\,e^{-m\operatorname{dist}(\operatorname{supp} X, \operatorname{supp} Y)}, \qquad (5)$$

uniformly in L.

Consequences of (2): Locality of the time evolution

Theorem (Locality of the time evolution, E.–Klein–Stolz '17)

Let X be a local observable. Then Eq. 2 implies that for all $t \in \mathbb{R}$ and $\ell > 0$ there exist constants $m' > 0, C < \infty$ and a local observable $X_\ell(t)$ with support contained in $[\min \operatorname{supp} X - \ell, \max \operatorname{supp} X + \ell]$, such that

$$\mathbb{E}\left(\sup_{t\in\mathbb{R}} \|P_{I_1}\left(X_\ell(t) - \tau_t\left(X\right)\right)P_{I_1}\|_1\right) \leq C\|X\|e^{-m'\ell}. \tag{6}$$

Consequences of (2): *Locality of the time evolution*

Theorem (Locality of the time evolution, E.–Klein–Stolz '17)

Let X *be a local observable. Then Eq. 2 implies that for all* $t \in \mathbb{R}$ *and* $\ell > 0$ *there exist constants* $m' > 0, C < \infty$ *and a local observable* $X_\ell(t)$ *with support contained in* $[\min \operatorname{supp} X - \ell, \max \operatorname{supp} X + \ell]$*, such that*

$$\mathbb{E}\left(\sup_{t\in\mathbb{R}} \|P_{I_1}\left(X_\ell(t) - \tau_t\left(X\right)\right) P_{I_1}\|_1\right) \leq C\|X\|e^{-m'\ell}. \tag{6}$$

- In addition, the uniform area law (in expectation) holds for eigenstates in the droplet regime (Beaud–Warzel '18).

Localization in XXZ model: Above the droplet spectrum

- Based on work in progress in collaboration with Abel Klein.

Combes-Thomas bound and failure of resolvent approach

Theorem (Combes-Thomas bound in I_1 sector)

Let ψ_E be an eigenvector of $H = H(\lambda = 0) + V$, where V is an arbitrary positive magnetic field and $E \in \sigma(H) \cap I_1$. Then one has the **deterministic** *bound*

$$\|\mathcal{N}_i \mathcal{N}_j \psi_E\| \leq C e^{-m|i-j|}, \tag{7}$$

that holds uniformly (in E).

- Cf. with (2), where one considers $\|\mathcal{N}_i \psi_E\| \, \|\mathcal{N}_j \psi_E\|$ on l.h.s. instead. Note that (7) holds even when $V = 0$ i.e. for translation invariant system (no localization).

Difficulty with the resolvent approach

- Spells trouble down the road if one attempts the resolvent approach to localization: For I_1 where a-posteriori we know that states resemble a single cluster of down spins, one can study $\|\mathcal{N}_i R_z \mathcal{N}_j\|$ where $R_z = (H-z)^{-1}$.
- However, for states that resemble say two clusters, the resolvent approach suggests to consider $\|\mathcal{N}_i \mathcal{N}_j R_z \mathcal{N}_k \mathcal{N}_m\|$.
- At best, the information one can extract from this object is one about $\|\mathcal{N}_i \mathcal{N}_j \psi_E\| \, \|\mathcal{N}_k \mathcal{N}_m \psi_E\|$...

Localization criterion in XXZ chain, I_2 sector

Definition

Let $\tau \in (0,1)$. We will call $\Lambda_L := [-L, L] + x$ a m_2-localizing box for H_ω in $I \subset I_2$ if for all normalized eigenfunctions ψ_λ with $\lambda \in \sigma(H_\omega^\Lambda) \cap I$ and all $R \in \mathbb{N}$ we can find either

- (A) $x_o \in \Lambda_L$ or
- (B) $(x_o, y_o) \in \Lambda_L \times \Lambda_L$ with $|x_o - y_o| \geq L^\tau$

such that the following bounds hold:

$$\|\mathcal{N}_i \psi_\lambda\| \leq \begin{cases} e^{-m_2|i-x_o|} & \text{if } (A) \text{ and } |i - x_o| > L^\tau \\ e^{-m_2 \operatorname{dist}(i,\{x_o,y_o\})} & \text{if } (B) \text{ and } \operatorname{dist}(i, \{x_o, y_o\}) > L^\tau \end{cases}, \tag{8}$$

and

$$\left\|P_+^{(J_R(x_o))}\psi_\lambda\right\| + \left\|P_+^{(J_R(y_o))}\psi_\lambda\right\| \leq e^{-m_2 h_I(\lambda) R}, \text{ for } R > L^\tau. \tag{9}$$

EMSA in XXZ chain

We can prove the following result:

Theorem (Localization in I_2 sector of random XXZ, E.–Klein '19+)

Fix $\delta > 0$. Then for λ large enough there exist constants $m_2 > 0, \beta > 0$ and a scale $L_0 > 0$ such that the probability that any Λ_L box is m_2-localizing box for H_ω exceeds $1 - e^{-L^\beta}$, provided $L \geq L_0$.

- The method of proof: the eigensystem multiscale analysis (EMSA).
- Previously developed for Anderson Hamiltonian (E.–Klein, '16–18).
- Requires proving (8) - (9) simultaneously (scale by scale).
- (9) and the Combes-Thomas bound below imply (a weaker version of) (8).
- While it is pretty clear how to go up in energy (to I_k), no chances of pushing it to $T = \infty$ regime since inter-cluster distance is no longer small. A completely different characterization of MBL is required...

Localization criterion in XXZ chain

Theorem (Combes-Thomas bound in I_k sector)

Let ψ_E be an eigenvector of $H = H(\lambda = 0) + V$, where V is an arbitrary positive magnetic field and $E \in \sigma(H) \cap I_k$. Then one has the **deterministic** *bound*

$$\left\| \prod_{n=1}^{k+1} \mathcal{N}_{i_n} \psi_E \right\| \leq C e^{-m(D(\{i_n\})-N)}, \tag{10}$$

where $D(\{i_n\})$ is the all-pairs shortest path, i.e., $D(\{i_n\}) = \min_{i_n \neq i_m} |i_n - i_m|$, and $\mathcal{N}\psi = N\psi$ (i.e., N is the total magnetization).

Consequences of (9)

- By itself, (9) **does not** imply exponential clustering.
- Computational basis $\beta := \otimes_{i=-L}^{L} e_i^{\alpha}$, $\alpha \in \{0,1\}$, $e_i^0 =\uparrow$, $e_i^1 =\downarrow$.
- Consider a state ψ of the form

$$\psi = e_{x_o}^1 \otimes e_{y_o}^1 \otimes_{i \neq x_o, y_o} e_i^0 + e_{x_o+1}^1 \otimes e_{y_o+1}^1 \otimes_{i \neq x_o+1, y_o+1} e_i^0 , \tag{11}$$

  Then it satisfies (9), but

$$\langle \psi, \mathcal{N}_{x_o} \mathcal{N}_{y_o+1} \psi \rangle = 0, \quad \langle \psi, \mathcal{N}_{x_o} \psi \rangle = \langle \psi, \mathcal{N}_{y_o+1} \psi \rangle = 1 .$$

- In fact, such ψ is a matrix product state, a generalization of the product state concept.

Consequences of (9)

- Not a fluke: The exponential clustering will fail in general if the eigenvalues in **droplet band** are not level spaced, in the appropriate sense.
- However, with level spacing, (9) does imply (dynamical) exponential clustering property,

$$R_{\tau_t(X),Y}(\psi_E) \leq C\,\|X\|\,\|Y\|\,e^{-m\,\mathrm{dist}(\mathrm{supp}\,X,\mathrm{supp}\,Y)}.$$

- Information propagation estimate similar to the one we have in the droplet regime does not hold: Even $\langle \psi_n, [\mathcal{N}_j, \tau_t(\mathcal{N}_i)]\,\psi_k \rangle$ does not decay with $|i-j|$ for $E_n, E_k \in I_2$, in general. However,

Theorem

Suppose that (9) *hold for all eigenvectors in* I_2. *Then for* $t \geq 1$ *there exists an operator* $\mathcal{O}_t$ *supported on* $[i-5\ell, i+5\ell]$ *such that*

$$\|P_{I_2}(\tau_t(\mathcal{N}_i) - \mathcal{O}_t)P_{I_2}\| \leq Cte^{-\ell/2}, \quad \tau_t(A) = e^{itH}Ae^{-itH}. \qquad (12)$$

Proofs ideas: Elements of EMSA

- To get a feeling for the eigensystem multiscale analysis, let's consider I_1 (the droplet spectrum).
- Let $E \in I_1$ and $J = [E - A, E + A]$ for some small ε. Let $h_J(s) = 1 - (s - E)^2/A^2$ for $s \in J$, $h_J(s) = 0$ otherwise.
- Suppose that some box $\Lambda_\ell(x) := [-\ell, \ell] + x$ in a big box Λ_L is m_1-localizing for H_ω, meaning (in this setting) that

Localizing box

For all normalized eigenfunctions ϕ_ν of $H_\omega^{\Lambda_\ell(x)}$ with $\nu \in \sigma(H_\omega^{\Lambda_\ell(x)}) \cap I_1$ there exists x_o s.t.

$$\|\mathcal{N}_i \phi_\lambda\| \leq e^{-m_1 h_J(\nu)|i - x_o|} \text{ provided } |i - x_o| > \ell^\tau. \tag{13}$$

- the factor $h_J(\nu)$ modulates localization length within the window of energies J.

Proofs ideas: Elements of EMSA

- Assume now that ψ_λ is an eigenvector of $H_\omega^{\Lambda_L}$ with $\lambda \in I_1$ and

No resonance condition

For some $\beta \in (0,1)$

$$\operatorname{dist}\left(\lambda, \sigma(H_\omega^{\Lambda_\ell(x)})\right) \geq e^{-m_1 \ell^\beta}. \tag{14}$$

Proofs ideas: Elements of EMSA

Lemma (Local decay for ψ_λ)

Assuming that (13) - (14) *hold, then there exists some* $y \in \Lambda_L \setminus \Lambda_\ell(x)$ *such that*

$$\|\mathcal{N}_x \psi_\lambda\| \leq e^{-m_1' h_J(\lambda)|x-y|} \|\mathcal{N}_y \psi_\lambda\|, \tag{15}$$

where m_1' *satisfies* $\frac{m_1'}{m_1} - 1 = O(\ell^{-\beta})$.

- Interpretation: Each time the spectrum associated with the Hamiltonian on a small scale ℓ is not too close to λ, we pick a decaying factor (at least $e^{-m_1 h_J(\lambda)\ell}$ small).
- It turns out that there exists (with large probability) *only one* small box $\Lambda_\ell(x_o)$ for which this can occur (a simple consequence of the so called Wegner between boxes estimate).
- This and the management of the bad boxes quickly leads to extension of (13) to the scale L.

Sketch of the proof

- The proof of the lemma above is based on locality property mentioned above and Gaussian filter functions.
- Specifically, let H_x denote the (decoupled Hamiltonian) $H_\omega^{\Lambda_\ell(x)} + H_\omega^{\Lambda_L \setminus \Lambda_\ell(x)}$. Note that the difference $\Delta H := H_\omega^{\Lambda_L} - H_x$ is supported at sites $x \pm \ell, x \pm (\ell + 1)$.
- We now decompose

$$\mathcal{N}_i \psi = \mathcal{N}_i \mathrm{e}^{-t\left((H_x - E)^2 - (\lambda - E)^2\right)} \psi + \mathcal{N}_i \left(1 - \mathrm{e}^{-t\left((H_x - E)^2 - (\lambda - E)^2\right)}\right) \psi.$$

The first term we split further as

$$\mathcal{N}_i \chi_J(H_x) \mathrm{e}^{-t\left((H_x - E)^2 - (\lambda - E)^2\right)} \psi + \mathcal{N}_i \chi_{J^c}(H_x) \mathrm{e}^{-t\left((H_x - E)^2 - (\lambda - E)^2\right)} \psi.$$

Sketch of the proof

- Let $\mathscr{S} \subset \Lambda_L$ and let $P_+^{\mathscr{S}}$ denote the projection onto the vacuum on $\mathscr{A}_{\mathscr{S}}$, and $P_-^{\mathscr{S}} := 1 - P_+^{\mathscr{S}}$. Then

$$\mathscr{N}_i \chi_{J^c}(H_x) \mathrm{e}^{-t\left((H_x-E)^2-(\lambda-E)^2\right)} \psi = \mathscr{N}_i \chi_J(H_x) \mathrm{e}^{-t\left((H_x-E)^2-(\lambda-E)^2\right)} P_-^{\Lambda_\ell(x)} \psi,$$

  and we can bound this term using

$$\left\| \chi_{J^c}(H_x) \mathrm{e}^{-t\left((H_x-E)^2-(\lambda-E)^2\right)} \right\| \leq e^{-tA^2 h_I(\lambda)}.$$

- To estimate $\mathscr{N}_i \left(1 - \mathrm{e}^{-t\left((H_i-E)^2-(\lambda-E)^2\right)}\right) \psi$, notice that

$$\begin{aligned} &\mathscr{N}_i \left(1 - \mathrm{e}^{-t\left((H_x-E)^2-(\lambda-E)^2\right)}\right) \psi \\ &= \mathscr{N}_i \left(1 - \mathrm{e}^{-t\left((H_x-E)^2-(\lambda-E)^2\right)}\right) (H_i - \lambda)^{-1} (H_x - H_L) \psi \end{aligned}$$

  where $F_{t,\lambda}$ is an analytic function. We use locality here.
- Finally, the term

$$\mathscr{N}_i \chi_J(H_x) \mathrm{e}^{-t\left((H_x-E)^2-(\lambda-E)^2\right)} \psi$$

  is estimated using localization of $\Lambda_\ell(x)$ box.

Proofs ideas: Locality

- Let $H = \sum_{n=-L}^{L-1} h_{n,n+1}$, $\text{supp}\,(h_{n,n+1}) = \{n, n+1\}$.
- Let $a < b$ be integers, and set $S = [-a, a]$, $T = [-b, b]$, $\ell = b - a$.
- Let $\hat{H} = H - h_{-b-1,-b} - h_{b,b+1}$ and assume that $[\hat{H}, P_+^{(T)}] = 0$ (this is the case for XXZ model).

Then $P_-^{(S)} e^{it\hat{H}} P_+^{(T)} = P_-^{(S)} P_+^{(T)} e^{it\hat{H}} = 0$ for all t. On the other hand, we have

$$\|e^{-itH} P_-^{(S)} e^{itH} - e^{-it\hat{H}} P_-^{(S)} e^{it\hat{H}}\| \leq \int_0^{|t|} \left\| \left[\left(H - \hat{H}\right), e^{-isH} P_-^{(S)} e^{isH} \right] \right\| ds$$

(exercise). By L–R, rhs is bounded by $C\,|t|\,e^{-m\ell}$ for all $|t| \leq \ell/v$, so

$$\begin{aligned}
\left\| P_-^{(S)} f(H) P_+^{(T)} \right\| &\leq \|\hat{f}\|_\infty \int_{|t| \leq \ell/v} \left\| P_-^{(S)} e^{itH} P_+^{(T)} \right\| dt + \int_{|t| > \ell/v} \left|\hat{f}(t)\right| dt \\
&\leq C \left\|\hat{f}\right\|_\infty \ell^2 e^{-\frac{1}{2}\ell} + \int_{v|t| > \ell} \left|\hat{f}(t)\right| dt. \quad \square
\end{aligned}$$

Proofs ideas: Combes–Thomas estimate

Let P be an orthogonal projection onto at most k cluster states, let $\bar{P} = 1 - P$. Let $E \in \sigma(H) \cap I_k$ and $H\psi = E\psi$. Then $\bar{P}(H-E)\bar{P} > \delta > 0$. Hence

$$0 = \bar{P}(H-E)\psi = \bar{P}(H-E)\bar{P}\psi + \bar{P}HP\psi,$$

which yields

$$\bar{P}\psi = -\left(\bar{P}(H-E)\bar{P}\right)^{-1}\bar{P}HP\psi.$$

Since $\left(\prod_{j=1}^{k+1} N_{i_j}\right)P = 0$ if $\{i_j\}$ are sufficiently spaced,

$$\left(\prod_{j=1}^{k+1} N_{i_j}\right)\psi = -\left(\prod_{j=1}^{k+1} N_{i_j}\right)\left(\bar{P}(H-E)\bar{P}\right)^{-1}\bar{P}HP\psi.$$

Note that $\bar{P}HP$ acts on the interface between k and $k+1$ cluster states. The graph distance between any particles' configuration with k clusters and the one where some particles are located at $\{i_j\}_{j=1}^{k+1}$ is at least $\ell - N$. This yields the desired decay via the variant of the Combes–Thomas estimate developed earlier in E.–Klein–Stolz. □

Thanks!

Time evolution $\tau_t^I(X)$

- Quasi-local observables: X is quasi-local centered at i if $\|[X, Y_j]\| \leq Ce^{-m|i-j|}$ for any Y_j with $\operatorname{supp} Y_j = \{j\}$.
- Quasi-local operators: T is quasi-local if $\|[[T, X], Y]\| \leq Ce^{-m \operatorname{dist}(\operatorname{supp} X, \operatorname{supp} Y)}$.
- In one particle setting, the operator $H_I = HP_I$ is quasi-local in the window I of localization, i.e.

$$|H_I(x, y)| \leq Ce^{-m|x-y|}.$$

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$$|H_I(x,y)| \leq Ce^{-m|x-y|}.$$

- This feature fails in the many body setting, as even H^2 is not local!
- Natural question: Why $\tau_t^I(X)$ in the dynamical clustering result works, and how it should be interpreted?
- What other consequences can be drawn from Eq. 2?

Consequences of (2)*: Zero velocity LR bounds*

Theorem (Zero velocity LR-type bounds, E.–Klein–Stolz '17)

Let X, Y and Z be local observables and let $I = I_1$. Set

$$J = \{0\} \cup I, \;\; X_I = P_I X P_I, \;\; X_0 = P_0 X P_0.$$

Then Eq. 2 implies that

1. $[X_0, Y_0] = 0, \;\; \mathbb{E}\left(\sup_{t\in\mathbb{R}} \left\| \left[\tau_t^I(X_I), Y_I\right] \right\|_1\right) \le C\|X\|\|Y\| e^{-\frac{1}{8} m \,\mathrm{dist}(X,Y)};$
2. $$\mathbb{E}\left(\sup_{t\in\mathbb{R}} \left\| \left[\tau_t^J(X_J), Y_J\right] - P_J\left(\tau_t^J(X) P_0 Y - Y P_0 \tau_t^J(X)\right) P_J \right\|_1\right) \le C\|X\|\|Y\| e^{-\frac{1}{8} m \,\mathrm{dist}(X,Y)};$$
3. $$\mathbb{E}\left(\sup_{t,s\in\mathbb{R}} \left\| \left[\left[\tau_t^J(X_J), \tau_s^I(Y_J)\right], Z_J\right] \right\|_1\right) \le C\|X\|\|Y\|\|Z\| e^{-\frac{1}{8} m \min\{\mathrm{dist}(X,Y),\mathrm{dist}(X,Z),\mathrm{dist}(Y,Z)\}}.$$

Locality in the droplet spectrum

Interpretation of (i)

The item (i) for $t = 0$ indicates that X_I is a quasi-local observable in the subspace of the single cluster eigenstates of H (and same with X_0 in the subspace of no clusters states (that happen to have dimension one). It also implies that H_I is the quasi-local operator in the same subspace. Thus $\tau_t^I(X_I)$ is the physically meaningful object.

Meaning of (ii)-(iii)

The item (ii) indicates that X_J is *not* a quasi-local operator as it mixes the single cluster eigenstates with no clusters states. In particular, it is not a good idea to select the subspace in which you want to prove LR bounds by spectral characteristics alone. But one can circumvent this difficulty by considering the higher order commutators, as in the item (iii).

Consequences of (9)

- Matrix product state (MPS) is a pure quantum state of many particles, written in the following form:

$$\psi = \sum_{\{s\}} \mathrm{Tr}[A_1^{(s_1)} A_2^{(s_2)} \cdots A_L^{(s_L)}] \otimes_{p=1}^{L} e_p^{s_p}, \tag{16}$$

where $s = (s_1, \ldots, s_L)$, s_i's take values $0, 1$ and $A_p^{(s_p)}$ are complex, square matrices of order d_p (called local dimension). For ψ in (11) one can choose

$$A_p^{(0)} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; \quad A_p^{(1)} = 0, \text{ for } p \neq i, i+1, j, j+1, \tag{17}$$

and

$$A_{i,j}^{(0)} = P; \quad A_{i,j}^{(1)} = \bar{P}, \quad A_{i+1,j+1}^{(0)} = \bar{P}; \quad A_{i+1,j+1}^{(1)} = P, \tag{18}$$

with

$$P = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}; \quad \bar{P} = 1 - P. \tag{19}$$