Large deviations of subgraph counts for sparse random graphs

Banff, Canada, August 7, 2019

Amir Dembo, Stanford University

Joint works with Nicholas Cook and with Sohom Bhattacharya

Celebrating HT Yau's 60-th birthday!

Universality for typical behavior: Examples

- **CLT:** for $X_1, X_2, \ldots$ iid, $\mathbb{E}\, X_1 = 0$, $\mathbb{E}\, X_1^2 = 1$,

$$\forall\ a < b, \quad \mathbb{P}\left\{ \tfrac{X_1 + \cdots + X_N}{\sqrt{N}} \in [a, b] \right\} \longrightarrow \gamma([a, b]) \qquad (\textit{universal}).$$

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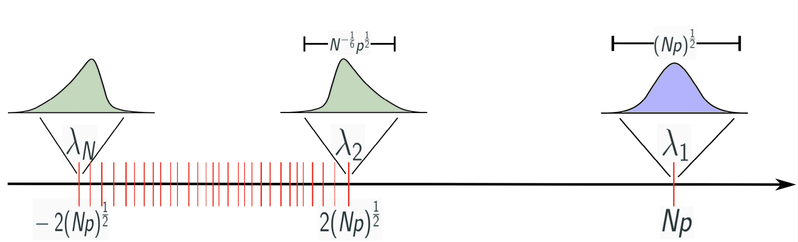

λ_1 is asymptotically Gaussian.

For $p \gg N^{-2/3}$: $\lambda_2, -\lambda_N$ follow the Tracy–Widom law [Lee–Schnelli '16].

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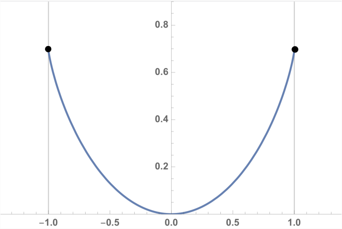
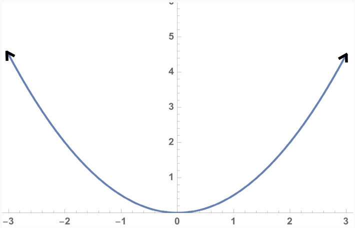


Rademacher

$\frac{1+x}{2} \log(1+x) + \frac{1-x}{2} \log(1-x)$



Gaussian

$\frac{x^2}{2}$

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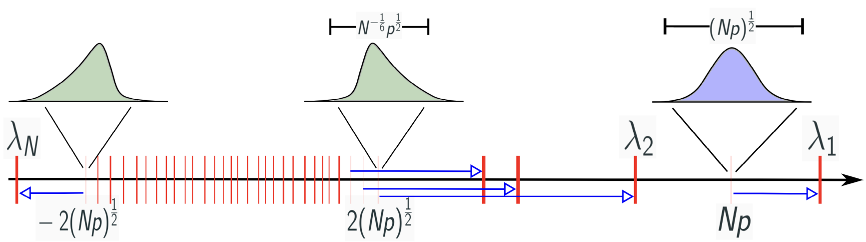

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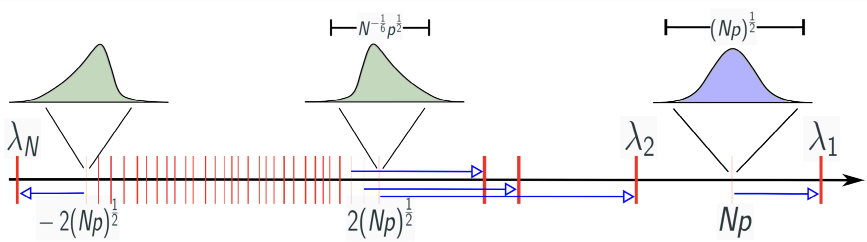

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(Tails for eigenvalues will be under the hood.)

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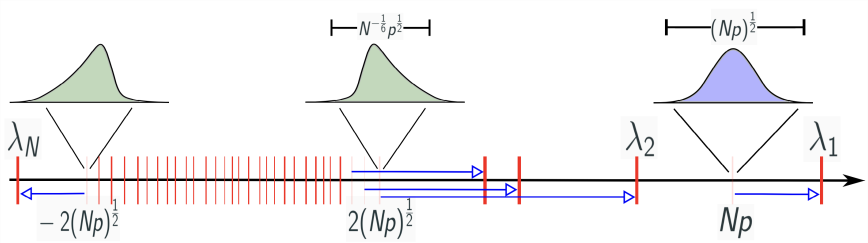

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Note we consider outliers at scale Np (for LDP at scale of the bulk cf. Guionnet–Husson 17' for $p = 1/2$).

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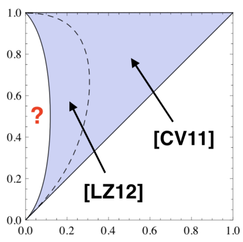
  Answer is (A) for much (not all!) of $0 < p < q < 1$ fixed.

  [Chatterjee–Varadhan '11]+[Lubetzky–Zhao '12].



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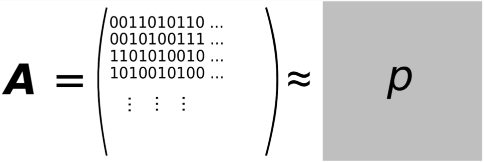

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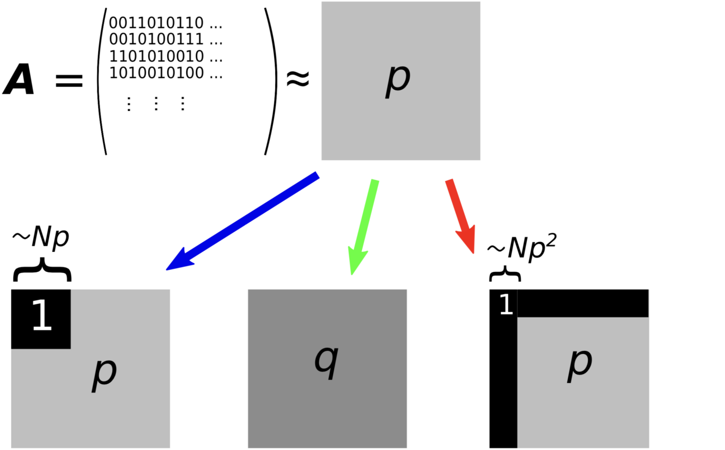

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- We also get:
  - ∗ lower tails (reduction to variational problem – can solve only for Sidorenko graphs);
  - ∗ upper tails for $\lambda_1, \lambda_2, -\lambda_N$ (together with subsequent work by [Bhattacharya–Ganguly '18] solving the LDP variational problem).

Previous approaches to upper tails

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- Obtain conditions for validity of the naïve mean field approximation:

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Previous approaches to upper tails

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- Disadvantage: Errors in the passage from indicator functions to smooth approximations cause a sub-optimal range of sparsity.

Dense case (Chatterjee–Varadhan '11)

- For a sequence of probability measures μ_N on a common topological space $\mathcal{X}$, large deviations principle (LDP) yields asymptotics of form

$$\mu_N(\mathcal{E}) \approx \exp\big(-v_N \inf_{x\in\mathcal{E}} J(x)\big), \qquad \mathcal{E} \subseteq \mathcal{X},$$

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- Graphons are limits of rescaled adjacency matrices, and $\|\cdot\|_\square$ extends the matrix cut-norm $\|M\|_\square = \max_{U,V\subseteq[N]} \left|\sum_{(i,j)\in U\times V} M_{ij}\right|$.

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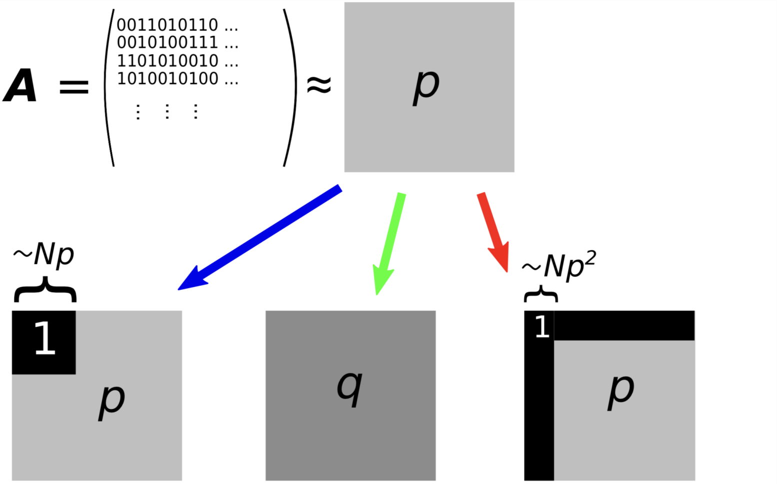
Identify a finite graph $G \in \mathcal{G}_N$ with $g \in \mathcal{W}$ via its adjacency matrix A, putting $g(x, y) := A_{\lfloor Nx \rfloor, \lfloor Ny \rfloor}$. General $g \in \mathcal{W}$ is like a "continuum adjacency matrix".



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Moral: the cut-norm topology is the right topology if you're interested in subgraph counts.

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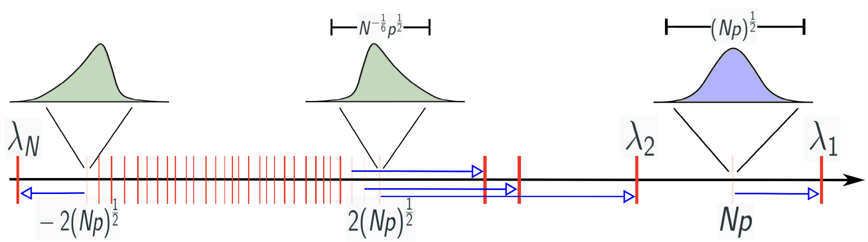
- Existing sparse graph limit theories, such as L^p-graphons [Borgs–Chayes–Cohn–Zhao '14], lack a strong enough counting lemma.
- We get much improved regularity and counting lemmas after cutting out appropriate small "bad" events (involving outlier eigenvalues).



Spectral regularity lemma for random graphs

Write $\mathcal{A}_N = \{0,1\}^{\binom{N}{2}}$ for the space of adjacency matrices and $\mathcal{X}_N = [0,1]^{\binom{N}{2}}$ for its convex hull (weighted adjacency matrices).

Proposition (Quantitative compactness for $\mathcal{A}_N$)

Let $N \in \mathbb{N}, K \geq 1$, $p \in (0,1)$ with $Np \geq \log N$, and $1 \leq R \leq Np$. There exists a partition $\mathcal{A}_N = \bigsqcup_{j=0}^{J} \mathcal{E}_j$ with the following properties:

(a) $\log J \lesssim RN \log(3 + \frac{R}{Kp})$;

(b) $\mathbb{P}\{\boldsymbol{A}_{N,p} \in \mathcal{E}_0\} \lesssim \exp(-cK^2N^2p^2)$;

(c) For each $1 \leq j \leq J$, there exists $Y_j \in \mathcal{X}_N$ of rank at most R such that $\|A - Y_j\|_{\mathrm{op}} \lesssim \frac{KNp}{\sqrt{R}}$ for all $A \in \mathcal{E}_j$.

Spectral counting lemma for random graphs

Proposition (Lipschitz continuity for homomorphism counts)

Let $H = (V, E)$ of max degree D.

Let $N \in \mathbb{N}$ and $p \in (0, 1)$. For $K \geq 1$ set

$$\mathcal{E}_H(K) = \left\{ X \in \mathcal{X}_N : \exists F \leq H \text{ with } \hom(F, X) > KN^{|V_F|} p^{|E_F|} \right\}.$$

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(b) For any $X, Y \in \mathcal{X}_N$ with $X \notin \mathcal{E}_H(K)$, for all $F \leq H$,

$$|\hom(F, X) - \hom(F, Y)| \lesssim_H KN^{|V_F|} p^{|E_F|} \frac{\|X - Y\|_{\mathrm{op}}}{Np^{D}}.$$

Beyond G(N,p)

Special properties of $G(N,p)$ and event $\{\mathcal{N}_H(\boldsymbol{G}) \geq t\}$:

- Independence (of edges)
- Homogeneity (exchangability, same p)
- One dimensional (one H)

Theorem (D.-Bhattacharya '19)

[Cook-D. '18] conclusions extend to:

- Uniform random graph $G^{(m)}(N)$, number of edges $m = \binom{N}{2}p$.
- Random d-regular graph $G^d(N)$, degree $d = Np$ (if H regular).
- $\mathbb{P}\left\{\mathcal{N}_{H_i}(\boldsymbol{G}) \geq (1+\delta_i)\,\mathbb{E}\mathcal{N}_{H_i}(\boldsymbol{G}), i \leq k\right\}$ joint upper tail.
- Inhomogeneous $G(N,\mathbf{p})$ as in Stochastic block model.

Semi-universal: [Cook-D. '18] reduction to variational problem is robust.
But [BGLZ '16] solution - special for $G(N,p)$; Re-done (change $c_H(\delta)$).

Thank you and

Many happy birthdays – HT!