Central Limit Theorem for the entanglement entropy of free disordered fermions

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based on the joint papers with

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Workshop "From Many Body Problems to Random Matrices"
BIRS, 4-9 August 2019

Anderson model

Random Schrödinger operator in $\mathbb{Z}^d$

$$(H\psi)(x) = V(x)\psi(x) + \sum_{x':|x'-x|=1} \psi(x'), \quad x, x' \in \mathbb{Z}^d, \psi \in l_2[\mathbb{Z}^d]$$

$\{V(x)\}_{x\in\mathbb{Z}^d}$-i.i.d. random variables

Spectral projections

Let $\mathcal{E}_H(\lambda)$ be a resolution of identity, and $P = \mathcal{E}_H(E)$ be its spectral projection corresponding the interval $I = (-\infty, E]$.

Large box entanglement entropy

Entanglement entropy

Consider a large box

$$\Lambda = [-L, L]^d, \quad P_\Lambda(x, y) = 1_\Lambda(x) P(x, y) 1_\Lambda(y)$$

Entanglement entropy, corresponding Λ

$$S_\Lambda = \mathrm{Tr}\,_\Lambda h(P_\Lambda)$$
$$h(t) = -t \log t - (1 - t) \log(1 - t), \quad t \in [0, 1]$$

We study the properties S_Λ, as $L \to \infty$, in particular:

$$E\{S_\Lambda\} \sim L^{m(d)}, \quad m(d)-?$$
$$\mathrm{Var}\{S_\Lambda\} \sim \quad L^{m_1(d)}, \quad m_1(d)-?$$
$$L^{-m_1(d)/2}(S_\Lambda - E\{S_\Lambda\}) \to \quad ?$$

Toy model of non interacting fermions

Consider a quadratic quantum Hamiltonian

$$\widehat{\mathrm{H}} = \sum_{\mathrm{x,y}\in\Omega} \mathrm{H(x,y)c^{+}(x)c^{-}(y)}, \quad \mathrm{x,y} \in \Omega = [-\mathrm{N},\mathrm{N}]^{\mathrm{d}}$$

where $\mathrm{c^{-}(x)}$, $\mathrm{c^{+}(x)}$ are the Fermi annihilation and creation operators

$$\{\mathrm{c^{-}(x), c^{+}(y)}\} = \delta_{\mathrm{xy}}$$

$\mathrm{H(x,y)}$ assumed to be self-adjoint operator $\mathrm{H(x,y)} = \overline{\mathrm{H(y,x)}}$.
Consider $\Lambda = [-\mathrm{L},\mathrm{L}]^{\mathrm{d}} \subset \Omega$

$$1 << \mathrm{L} << \mathrm{N}$$

Then $\hat{\mathrm{H}}$ acts in $\mathcal{H}(\Lambda) \otimes \mathcal{H}(\Omega \setminus \Lambda)$. If we consider the density matrix $\hat{\rho}$ of $\widehat{\mathrm{H}}$ and set

$$\rho_\Lambda = \mathrm{Tr}_{\,\Omega\setminus\Lambda}(\rho),$$

then entanglement entropy of Λ is

$$\mathrm{S}_\Lambda = \lim_{\mathrm{N}\to\infty} \mathrm{Tr}\, \rho_\Lambda \log_2 \rho_\Lambda.$$

Link with Szegö's theorem

Determinant of the Toeplitz matrix

Consider an infinite Toeplitz matrix in $d = 1$ case

$$A_{jk} = A_{j-k,0} = A_{j-k}, \quad A = a(H_0),$$

where H_0 is a discrete Laplace operator.
We restrict A on the interval $\Lambda = [-L, L]$

$$A^{(L)} = 1_{[-L,L]} A 1_{[-L,L]} = 1_{[-L,L]} a(H_0) 1_{[-L,L]} = a_\Lambda(H_0)$$

and consider

$$\log \det A^{(L)} = \operatorname{Tr} \log a_\Lambda(H_0)$$

The same happens in $d > 1$ case.

Hence the logarithm of the Toeplitz determinant is some special case of the functional of operator, which is determined by two functions a and φ and by the cube Λ

$$\Psi_\Lambda[H_0; a, \varphi] = \operatorname{Tr} \varphi(a_\Lambda(H_0)).$$

Szegö's theorem

Under rather general assumptions (when a and φ are e.g. C_1)

$$\Psi_\Lambda[H_0; a, \varphi] = L^d C_0(\nu) + L^{d-1} C_1(a, \varphi) + o(L^{d-1})$$

where

$$\nu(x) = \varphi(a(x))$$

The first term is proportional to the volume of Λ (volume term) and the second is proportional to the area of the faces of Λ (area term).
If a or φ have a finite number of jumps, then

$$\Psi_\Lambda[H_0; a, \varphi] = L^d C_0(\nu) + L^{d-1} \log L C_1'(a, \varphi) + o(L^{d-1} \log L).$$

Here we have the violation of the area law.

Results on the asymptotic of entanglement entropy can be treated as a stochastic analogue of Szegö's theorem, when

$$a(x) = 1_{(-\infty,E]}(x), \quad \varphi(x) = h(x).$$

Remark that in this case

$$\nu(x) = 0.$$

The most general setting

Let H be a random Schrödinger operator with i.i.d. potential.

Given two functions a and φ we want to study the asymptotic behaviour of the functional

$$\Psi_\Lambda[\mathrm{H}; \mathrm{a}, \varphi] = \mathrm{Tr}\, \varphi(\mathrm{a}_\Lambda(\mathrm{H}))$$

in the limit $\mathrm{L} \to \infty$.

The simplest case $a(x) = x$. Law of Large Numbers

In this case we have

$$\Psi_\Lambda[H; a, \varphi] = \operatorname{Tr}\varphi(H_\Lambda) = \sum \varphi(\lambda_i(H_\Lambda)) = \mathcal{N}_\Lambda[\varphi],$$

where $\mathcal{N}_\Lambda[\varphi]$ is a linear eigenvalue statistics of H_Λ.

It is well known that there exists a measure σ, such that we have a volume law

$$|\Lambda|^{-1} E\{\mathcal{N}_\Lambda[\varphi]\} \to \int \varphi(x) d\sigma(x), \text{ as } L \to \infty.$$

We have also self averaging property

$$\operatorname{Var}\{|\Lambda|^{-1}\mathcal{N}_\Lambda[\varphi]\} \to 0, \text{ as } L \to \infty.$$

The simplest case a(x) = x. CLT

Sobolev space $\mathcal{H}_\alpha$

We say the $\varphi \in \mathcal{H}_\alpha$ if

$$||\varphi||_\alpha^2 = \int (1+2|\mathrm{k}|)^{2\alpha} |\widehat{\varphi}(\mathrm{k})|^2 d\mathrm{k}, \quad \widehat{\varphi}(\mathrm{k}) = \frac{1}{2\pi} \int e^{i\mathrm{kx}} \varphi(\mathrm{x}) d\mathrm{x}.$$

Theorem

If $\varphi \in \mathcal{H}_\alpha$ with $\alpha > 1$, then

$$|\Lambda|^{-1/2} \big(\mathcal{N}_\Lambda[\varphi] - \mathrm{E}\{\mathcal{N}_\Lambda[\varphi]\} \big) \to (\mathcal{V}\varphi, \varphi)^{1/2} \mathcal{N}(0,1), \text{ as } \mathrm{L} \to \infty,$$

where $\mathcal{V}$ is non negative bounded operator in $\mathcal{H}_\alpha$.

Case of smooth a, φ

Theorem[Large Numbers Law][Pastur,S:18]

Let $\mathrm{a} \in \mathcal{H}_\theta$, $\theta > (\mathrm{d}+1)/2$, and $\nu \in \mathcal{H}_\alpha$ with $\alpha > 1$ ($\nu(\mathrm{x}) = \varphi(\mathrm{a}(\mathrm{x}))$), then there exists a measure σ, such that we have a volume law

$$|\Lambda|^{-1}\mathrm{E}\{\Psi_\Lambda[\mathrm{H};\mathrm{a},\varphi]\} \to \int \nu(\lambda)d\sigma(\lambda), \text{ as } \mathrm{L} \to \infty.$$

Theorem[CLT for smooth case][Pastur,S:18]

If $\mathrm{a} \in \mathcal{H}_\theta$, $\theta > (\mathrm{d}+1)/2$ and $\nu \in \mathcal{H}_\alpha$ with $\alpha > 1$, then

$$|\Lambda|^{-1/2}(\Psi_\Lambda[\mathrm{H};\mathrm{a},\varphi] - \mathrm{E}\{\Psi_\Lambda[\mathrm{H};\mathrm{a},\varphi]\}) \to (\mathcal{V}\nu,\nu)_\alpha^{1/2}\mathcal{N}(0,1), \text{ as } \mathrm{L} \to \infty.$$

Case of $a = 1_{(-\infty,E]}$, $\varphi = h$

Recall that in this case

$$\nu(x) = h(a(x)) = 0,$$

so no hope to use previous results directly.

Observe that $h(t)$ is symmetric with respect to $x = 1/2$

$$h(1/2 - t) = h(1/2 + t).$$

Hence there is an increasing function h_0 defined on $(0, 1/4)$ such that

$$h(t) = h_0(x(t)), \quad x(t) = t(1-t), \quad x \in [0, 1/4]$$
$$\Leftrightarrow h_0(x) = h(t(x)), \quad t(x) = \frac{1}{2}(1 - \sqrt{1-4x})$$

It is easy to check that $h_0'(x) \to 2$, as $x \to 1/4$. Hence we can extend $h_0(x)$ to $\mathbb{R}$ in such a way that $h_0 \in \mathcal{H}_{3/2-\varepsilon}$.
Since $0 \le P_\Lambda \le 1$ it is evident that for any such extension

$$\mathrm{Tr}\, h(P_\Lambda) = \mathrm{Tr}\, h_0(P_\Lambda(1 - P_\Lambda))$$

New setting

Operator Π_Λ

$$\Pi_\Lambda = P_\Lambda(1 - P_\Lambda); \quad \Pi_\Lambda(x, y) = \sum_{z \notin \Lambda} P(x, z)P(y, z), \quad x, y \in \Lambda$$

Linear eigenvalue statistics of Π_Λ

$$\mathcal{N}_\Lambda[\varphi; \Pi_\Lambda] = \mathrm{Tr}\, \varphi(\Pi_\Lambda), \quad \mathcal{N}_\Lambda^\circ[\varphi; \Pi_\Lambda] = \mathcal{N}_\Lambda[\varphi; \Pi_\Lambda] - \mathrm{E}\{\mathcal{N}_\Lambda[\varphi; \Pi_\Lambda]\}$$

We study the behaviour of the random variable $\mathcal{N}_\Lambda[\varphi]$, as $\Lambda \to \infty$.
The same questions:

$$\mathrm{E}\{\mathcal{N}_\Lambda[\varphi; \Pi_\Lambda]\} \sim L^{m(d)}\phi_0, \quad m(d)-?$$
$$\mathrm{Var}\{\mathcal{N}_\Lambda[\varphi; \Pi_\Lambda]\} \sim L^{m_1(d)} \quad m_1(d)-?$$
$$L^{-m_1(d)/2}\mathcal{N}_\Lambda^\circ[\varphi; \Pi_\Lambda] \to \quad ?$$

Localization assumptions

Our main technical assumption is that the so-called fraction moment criteria for the Anderson localization is fulfilled, i.e. for some $s < 1$

$$E\{|(H - E - i\varepsilon)^{-1}(x, y)|^s\} \le C(s)e^{-c(s)|x-y|} \quad (1)$$

The assumption implies, in particular, a very important bound

$$E\{|P(x, y)|\} \le Ce^{-c|x-y|}$$

It is known (see e.g. the paper of Aizenman, Schenker, Friedrich, and Hundertmark (CMP, 01)), that if (1) is fulfilled for E of some interval (E_1, E_2), then the spectrum H in (E_1, E_2) is pure point, end the eigenvectors are localized (their components decay exponentially).

When does criteria (1) fulfill?

- E belongs to the spectral gap of H;
- any $E \in \sigma(H)$, $d = 1$ and i.i.d. potentials (Minami 96);
- any $E \in \sigma(H)$, $d > 1$ and $V(x)$ has a sufficiently large amplitude (Aizenman-Molchanov 93);
- E belongs to a neighbourhood of the spectrum edges, $d > 1$, and $V(x)$ has any amplitude (Aizenman-Graf 98);

Case of d = 1

Theorem[Elgart, Pastur, S:17]

There exists

$$\lim_{L\to\infty} E\{\mathcal{N}_\Lambda[h_0]\}.$$

The result corresponds to the "area law" for d = 1.

Theorem[Pastur:16]

Large block entanglement entropy for d = 1 does not possess the self averaging property:

$$\lim_{L\to\infty} \mathrm{Var}\{\mathcal{N}_\Lambda[h_0]\} \neq 0$$

Case of $d \geq 2$: LLN

Theorem[Elgart, Pastur, S:17]

Let the Anderson localization criteria (1) is fulfilled. Then there exists

$$\lim_{L\to\infty} E\{L^{-(d-1)}\mathcal{N}_\Lambda[h_0]\}.$$

The result corresponds to the "area law" for $d \geq 2$.

Theorem[Elgart, Pastur, S:17]

If the Anderson localization criteria (1) is fulfilled, then the large block entanglement entropy for $d \geq 2$ possesses the self averaging property:

$$\lim_{L\to\infty} \mathrm{Var}\{L^{-(d-1)}\mathcal{N}_\Lambda[h_0]\} = 0$$

Case of d $\geq$ 2: CLT

Theorem[Pastur, S:19]

Let $\varphi \in \mathcal{H}_\alpha$ with $\alpha > 1$. If the Anderson localization criteria (1) is fulfilled, then

$$\mathrm{L}^{-(\mathrm{d}-1)/2}\mathcal{N}_\Lambda^\circ[\varphi;\Pi_\Lambda] \to (\mathcal{V}\varphi,\varphi)_\alpha^{1/2}\mathcal{N}(0,1), \text{ as } \mathrm{L}\to\infty,$$

where $\mathcal{V}$ is non negative bounded in $\mathcal{H}_\alpha$ operator .

Scheme of the proof of CLT in the case of $\varphi(\Pi)$ and $\nu(\mathrm{H})$ (with smooth a)

CLT for martingales (modification of [Billingsly:95])

Let $X_k = E_{<k}\{Y - E_k Y\}$ be a martingale differences array with respect to independent random vectors $V_1, \ldots, V_n$, $S_n = \sum_{k=1}^n X_k$, $\sigma_n = \sum_{k=1}^n E\{X_k^2\} = O(1)$. Assume that

$$(1) \quad \sum E\{X_k^4\} \le \varepsilon_n, \qquad (2) \quad \mathrm{Var}\Big\{\sum_{k=1}^n X_k^2\Big\} \le \tilde{\varepsilon}_n$$

Then

$$|E\{e^{itS_n}\} - e^{-t^2\sigma_n/2}| \le C'(t)(\varepsilon_n^{1/2} + \tilde{\varepsilon}_n^{1/2}).$$

At the first step we use the theorem to prove CLT for the test functions of the form

$$\varphi_\eta = \varphi * \mathcal{P}_\eta$$

where $\mathcal{P}_\eta$ is the Poisson kernel

$$\mathcal{P}_\eta(\mathrm{x}) = \frac{1}{\pi}\frac{\eta}{\mathrm{x}^2+\eta^2}$$

It is easy to see that

$$\mathcal{N}_\Lambda[\varphi_\eta] = \pi^{-1}\int \varphi(\lambda)\Im\mathrm{Tr}\,\gamma(\lambda+\mathrm{i}\eta)\mathrm{d}\lambda.$$

where

$$\gamma(\mathrm{z}) = \begin{cases} \mathrm{Tr}(\mathrm{H}-\mathrm{z})^{-1}, & \text{for smooth a (i)} \\ \mathrm{Tr}(\Pi-\mathrm{z})^{-1} & \text{for a(H) = P (ii)} \end{cases}$$

Introduce

$$X_u(z) = L^{-l(d)/2}(\gamma(z) - \gamma_u(z)) \quad \text{with} \quad l(d) = d \text{ or } l(d) = d - 1$$

where γ_u is the trace of the resolvent of H_u or Π_u, where H_u is obtained by the replacing u-th line and column of H by 0, and Π_u is constructed from the spectral projection of H_u.

It is easy to check that in both cases it suffices to check 2 conditions:

$$(1) \qquad \sum_u E\{|X_u|^4\} \to 0$$

$$(2) \qquad \mathrm{Var}\Big\{\sum_u (\Im X_u)^2\Big\} \to 0$$

Checking these 2 conditions for z with $\Im z = \varepsilon$ we prove CLT for the functions $\varphi_\eta = \varphi * \mathcal{P}_\eta$.

Extension of CLT to $\varphi \in \mathcal{H}_\alpha$

Proposition 1

Let $\{\xi_l^{(n)}\}_{l=1}^n$ be a triangular array of random variables,

$$\mathcal{N}_n[\varphi] = \sum_{l=1}^{n} \varphi(\xi_l^{(n)})$$

be its linear statistics, corresponding to a test function $\varphi : \mathbb{R} \to \mathbb{R}$, and

$$V_n[\varphi] = \mathrm{Var}\{d_n^{-1/2} \mathcal{N}_n[\varphi]\}$$

be the variance of $\mathcal{N}_n[\varphi]$, where $\{d_n\}_{n=1}^\infty$ is some bounded from below sequence of numbers. Assume that
(a) there exists a space $\mathcal{L}$ with a norm $||...||$ such that for $\varphi \in \mathcal{L}$

$$V_n[\varphi] \le C||\varphi||^2, \ \forall \varphi \in \mathcal{L};$$

(b) there exists a dense subset $\mathcal{L}_1 \subset \mathcal{L}$ such that the CLT is valid for $d_n^{-1/2} \mathcal{N}_n[\varphi], \ \varphi \in \mathcal{L}_1$,
Then CLT is valid for all $d_n^{-1/2} \mathcal{N}_n[\varphi]$, $\varphi \in \mathcal{L}$.

Uniform bounds for the variance of LES

Proposition 3 [S:11]

For any real symmetric or hermitian matrix M with random entries, any $\alpha > 0$, and $\varphi \in \mathcal{H}_\alpha$ we have

$$\mathrm{Var}\{\mathrm{Tr}\,\varphi(\mathrm{M})\} \le \mathrm{C}_\alpha ||\varphi||_\alpha^2 \int_0^\infty \mathrm{dy} e^{-\mathrm{y}} \mathrm{y}^{2\alpha-1} \int_{-\infty}^\infty \mathrm{Var}\{\gamma(\mathrm{x}+\mathrm{iy})\}\mathrm{dx},$$
$$\gamma(\mathrm{z}) = \mathrm{Tr}\,(\mathrm{M}-\mathrm{z})^{-1}$$

Remark

Proposition 3 is more efficient than Helffer-Sjöstrand's formula, since, e.g., for Wigner and sample covariance matrices the formula requires φ to be C^3 function, while Proposition 3 requires $\varphi \in \mathrm{H}_\alpha$ with $\alpha > 2$.

Bounds for the variance of the resolvent trace

Proposition 2

In both cases (i) and (ii) for any $z : \Im z > 0$ there exists some $C > 0$ such that

$$L^{-l(d)} \mathrm{Var}\{\gamma(z)\} \le C \log^m |\Im z|^{-1} / |\Im z|^2,$$

where $l(d) = d$ for the case (i), $l(d) = d - 1$ for the case (ii) and m is some constant which is not important for us.