On 2-dimensional stochastic sine Gordon equation

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The equation

Hyperbolic sine-Gordon equation on $\mathbb{T}^2$ with space-time white noise forcing

$$\begin{cases} \partial_{tt}^2 u + (1-\Delta)u + \gamma \sin(\beta u) = \xi \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) \end{cases}$$

Here:

- $(t, x) \in \mathbb{R} \times \mathbb{T}^2 \approx \mathbb{R} \times [0, 2\pi)^2$.
- β, $\gamma \in \mathbb{R}$.
- ξ is *space-time white noise*, $\xi = \mathrm{d}W$ where W is a *cylindrical Brownian motion*.

Space-time white noise

Formally, $\xi = \mathrm{d}W$, with W *cylindrical Wiener process*,

$$W(t) := \sum_{n \in \mathbb{Z}^2} B_n(t) e_n,$$

$$e_n(x) := \frac{1}{2\pi} e^{in \cdot x},$$

$B_n(t)$ complex Brownian motions, independent except for the condition

$$B_{-n}(t) = \bar{B}_n(t).$$

Sine Gordon as PDE

The deterministic sine-Gordon and sinh-Gordon equations

$$\partial_{tt}^2 u - \partial_{xx}^2 u = \begin{cases} \sin u \\ \sinh u \end{cases} .$$

in 1 dimension have an integrable structure. (On $\mathbb{T}$: McKean, 1981)

Following Friedlander (1984), McKean (1993) constructs Gibbs type invariant measures for wave equations of the form

$$\partial_{tt} u - \partial_{xx}^2 u = f'(u),$$

including sine-Gordon.

Stochastic wave equations

- Large literature with **colored noise**, using stochastic analysis methods. Dalang, Mueller, Walsh, Ondrejat (2010)...
- With **white noise**. In dimension $d = 1$, Carmona-Nualart (1988) consider and general smooth nolinearity.
  They use the explicit Duhamel formula and Walsh's theory of 2 parameter martingales
- In dimension $d = 2$ Albeverio et al. construct solutions using Colombeau algebras. Gubinelli-Koch-Oh $d = 2, 3$ (2017, 2018) and Gubinelli-Koch-Oh-Tolomeo $d = 2$ use harmonic analysis methods to solve the equation with polynomial nonlinearity.

Stochastic Sine Gordon

Parabolic equation

$$\partial_t u + (1 - \Delta)u + \gamma \sin(\beta u) = \xi.$$

This is an invariant dynamics for continuum SG measure from Roland's talk

$$\frac{1}{Z} \exp\Big(- \frac{1}{2} \int_{\mathbb{T}^2} |\nabla u|^2 \, \mathrm{d}x + \frac{1}{2} \int_{\mathbb{T}^2} u^2 \, \mathrm{d}x + \frac{\gamma}{\beta} \int_{\mathbb{T}^2} : \cos(\beta u(x)) : \, \mathrm{d}x \Big).$$

Hairer and Shen (2014) obtain local in time solutions to this equation for $\beta^2 \in (0, \frac{16\pi}{3})$. Chandra, Hairer, Shen (2016) extend this to $\beta^2 < 8\pi$, expected to be the optimal threshold.

Thresholds

It is known from work on the static sine-Gordon measure (see introduction to Roland's paper!) that there is an infinite sequence of thresholds

$$\beta_n = \frac{8\pi n}{n+1}$$

where new divergent quantities appear and require renormalization. For Chandra, Hairer and Shen, this translates into the equation requiring further renormalizations to obtain local solutions.

In contrast, we can obtain local in time solutions for any $\beta > 0$.

Singularity

For zero initial data, first Picard iterate solves

$$\partial_{tt}^2 \Psi + (1-\Delta)\Psi = \xi.$$

Solution

$$\int_0^t \frac{\sin((t-t')\langle\nabla\rangle)}{\langle\nabla\rangle} dW(t') = \sum_{n\in\mathbb{Z}^2} \int_0^t \frac{\sin((t-t')\langle n\rangle)}{\langle n\rangle} \mathrm{d}B_n(t')\ e_n.$$

Here

$$\mathcal{F}\Big(\frac{\sin(t\langle\nabla\rangle)}{\langle\nabla\rangle} f\Big)(n) = \frac{\sin(t\sqrt{1+|n|^2})}{\sqrt{1+|n|^2}} \widehat{f}(n).$$

Variance of Fourier coefficient labelled by n

$$\mathbb{E}\left[\mathcal{F}(\Psi)(n)^2\right] = \frac{1}{4\pi^2}\int_0^t \Big(\frac{\sin((t-t')\langle n\rangle)}{\langle n\rangle}\Big)^2 \,\mathrm{d}t'$$
$$= \frac{1}{4\pi^2}\Big(\frac{t}{2\langle n\rangle^2} - \frac{\sin(2t\langle n\rangle^3)}{4\langle n\rangle^3}\Big).$$

Sum diverges logarithmically $\rightarrow \Psi$ is not a L^2 function.

Cannot define the nonlinearity

$$\sin(\beta\Psi) = \Im(e^{i\beta\Psi})$$

appearing in the next iterate.

Wick renormalization

For a Gaussian random variable $X \sim N(0, \sigma^2)$, define

$$: X^k : \stackrel{\text{def}}{=} H_k(X; \sigma),$$

where H_k is the kth Hermite polynomial:

$$e^{\lambda x - \frac{\sigma^2}{2}\lambda} = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} H_k(x; \sigma).$$

Define

$$: e^{iX} : \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} \frac{(i\beta)^k}{k!} : X^k := e^{\frac{\beta^2}{2}} e^{i\beta X}.$$

Renormalizing Ψ

Consider Fourier truncation of Ψ_N

$$\Psi_N = \sum_{n\in\mathbb{Z}^2} \chi_N(n) \int_0^t \frac{\sin((t-t')\langle\nabla\rangle)}{\langle\nabla\rangle} \, \mathrm{d}B_n(t')\, e_n,$$

$\chi_N(\cdot) = \chi(\cdot/N)$ smooth cutoff.

Variance

$$\sigma_N(t) = \mathbb{E}[\Psi_N(t,x)^2] \sim t\log N.$$

We show that

$$: e^{i\beta\Psi_N} : \stackrel{\text{def}}{=} e^{\frac{\beta^2}{2}\sigma_N^2(t)} e^{i\beta\Psi_N}.$$

converges in some function space and define $e^{i\beta\Psi}$ as its limit.

Da Prato-Debussche trick

Look for solutions of form

$$u_N = \Psi_N + v_N.$$

Residual term v_N satisfies the equation

$$\begin{cases} \partial_{tt}^2 v_N + (1-\Delta)v_N + \Im(: e^{i\beta\Psi_N} : e^{i\beta v_N}) = 0 \\ (v_N, \partial_t v_N)|_{t=0} = (u_0, u_1). \end{cases}$$

With renormalization, the terms $: e^{i\beta\Psi_N} :$ have a limit. We try to solve for v_n, in particular, we need to make sense of the product $: e^{i\beta\Psi_N} : e^{i\beta v_N}$.

This type of ansatz has appeared in McKean (1995), Bourgain (1996), Da Prato-Debussche (2002), Hairer (2014), etc.

Results: Local existence

We show the solutions to

$$\begin{cases} \partial_{tt}^2 u_N + (1-\Delta)u_N + \gamma_N \sin(\beta u_N) = \underbrace{P_N \xi}_{\text{Fourier truncation}} \\ (u_N, \partial_t u_N)|_{t=0} = (u_0, u_1) \end{cases}$$

with

$$\gamma_N(t,\beta) = e^{\frac{\beta^2}{2}\sigma_N(t)} \to \infty$$

converge in probability in a suitable space of distributions.

Results: Local existence

Theorem (Oh, Robert, S., Wang, 2019)

Let $\beta \neq 0$, *and* $s > 0$. *For any* $(u_0, u_1) \in H^s \times H^{s-1}$, *there exists* $T_0(\|u_0\|_{H^s}, \|u_1\|_{H^{s-1}})$ *such that for any* $T \leq T_0$, $\exists \Omega_N(T)$ *such that*

1. *for* $\omega \in \Omega_N(T)$, *there exists a unique* u_N *to the truncated SSG in form*

   $$\Psi_N + v_N \subset C([0,T], H^{-\epsilon}(\mathbb{T}^2)).$$

   v_N *has positive regularity.*

2. $$\mathbb{P}(\Omega_N(T)^c) \to 0$$

   uniformly in N *as* $T \to 0$.

Results: local existence

There exists a stopping time τ and a stochastic process u in $C([0,T];H^{-\epsilon}(\mathbb{T}^2))$, of the form

$$u = \Psi + v,$$

where v has positive regularity, such that, for each $T > 0$, u_N converges in probability to u in $C([0,T];H^{-\epsilon}(\mathbb{T}^2))$ on $\{\tau \geq T\}$.

Results: Triviality

Theorem (Oh, Robert, S., Wang, 2019)

Let $\beta \in \mathbb{R} \setminus \{0\}$ *and fix* $(u_0, u_1) \in H^s \times H^{s-1}$ *for some* $s > 0$*. Given any small* $T > 0$*, the solutions to the* **non-renormalized SSG equation**

$$\begin{cases} \partial_{tt}^2 u_N + (1-\Delta)u_N + \sin(\beta u_N) & = P_N \xi \\ (u_N, \partial_t u_N)|_{t=0} & = (u_0, u_1) \end{cases}$$

converge in probability to the solution of the linear stochastic wave equation

$$\begin{cases} \partial_{tt}^2 u + (1-\Delta)u & = \xi \\ (u, \partial_t u)|_{t=0} & = (u_0, u_1) \end{cases},$$

in $C([0,T], H^{-\epsilon}(\mathbb{T}^2))$*,* $\epsilon > 0$ *as* $N \to \infty$*.*

GKO

Gubinelli, Koch, Oh (2017) study solutions of the nonlinear wave equation in 2d:

$$\partial_{tt} u_N + (1 - \Delta) u_N \pm : u_N^k := P_N \xi,$$

with $k \geq 2$ an integer.

They show convergence in probability of the u_N in $C([0,T], H^{-\epsilon})$.

Renormalization is simpler, because the nonlinearity is of power type.

Chandra-Hairer-Shen

Hairer-Shen (2014) and Chandra-Hairer-Shen consider the equations

$$\partial_t u_N + (1 - \partial_x^2) : u_N : + \gamma : \sin(\beta u_N) := P_N \xi$$

and show that for $\beta^2 < 8\pi$, the solutions converge in some negative Hölder space to a limiting stochastic process.

Regularity of nonlinearity

Equation for $v_N = \Psi_N - u_N$

$$\partial_{tt}^2 v_N + (1-\Delta)v_N + \Im(: e^{i\beta\Psi_N} : e^{i\beta v_N}).$$

Rewrite as

$$v_N(t) = \partial_t S(t)u_0 + S(t)u_1 - \int_0^t S(t-t')\Im(: e^{i\beta\Psi_N} : e^{i\beta v_N})\mathrm{d}t',$$

$$S(t) = \frac{\sin(t\langle\nabla\rangle)}{\langle\nabla\rangle}.$$

We would like to control v_N a priori in Sobolev space H^s, $s > 0$.

To proceed, need to investigate regularity of $: e^{i\beta\Psi_N} :$.

Regularity of nonlinearity

Let $\alpha > 0$. Expand the exponential and use the identities

$$\mathbb{E}[\Psi_N(t,x_1)\Psi_N(t,x_2)] = \sum_{|n|\leq N} e_n(x_1 - x_2) \int_0^t \frac{\sin((t-s)\langle n\rangle)^2}{\langle n\rangle^2}\,\mathrm{d}s,$$

find

$$\begin{aligned} &\mathbb{E}[\|\langle\nabla\rangle^{-\alpha} : e^{i\beta\Psi_N} : \|_{L^2(\mathbb{T})}^2] \\ \leq &\sum_{k\geq 0} \sum_{|n_1|,\dots,|n_k|\leq N} \iint \langle n_1 + \cdots + n_k\rangle^{-2\gamma} \langle n_1\rangle^{-2} \cdots \langle n_k\rangle^{-2} \\ \lesssim &\, C(\beta^2, \alpha) < \infty, \end{aligned}$$

independently of N.

Higher moments

We will need higher moments of $\langle \nabla \rangle^{-\gamma} : e^{i\beta\Psi_N} :$ to close a fixed point argument.

For power type nonlinearity, we can control higher powers of the second moment, since

$$\mathbb{E}[|P_k(\phi)|^p] \lesssim p^{kp/2}\mathbb{E}[|P_k(\phi)|^2]^{p/2},$$

if ϕ is Gaussian and P_k has degree k.

Regularity of : $e^{i\beta\Psi_N}$:

$W^{\gamma,p}(\mathbb{T}^2)$ is defined by the norm

$$\|f\|_{W^{\gamma,p}(\mathbb{T})} = \|\langle\nabla\rangle^{\gamma} f\|_{L^p(\mathbb{T}^2)}.$$

Theorem

Let $\beta \neq 0$ and $\beta^2 T < 8\pi$. Given any $1 \leq p, q < \infty$ and $\alpha > \frac{\beta^2 T}{8\pi}$, the sequence of random variables : $e^{i\beta\Psi_N}$: is Cauchy in $L^p(\Omega; L^q([0,T]; W^{-\alpha,\infty}(\mathbb{T}^2)))$.

$W^{-\alpha,p}$ norm of $: e^{i\beta\psi_N} :$

Goal: compute $\| : e^{i\beta\Psi_N} : \|_{L^{2p}_\omega L^q_T W^{-\alpha,\infty}_x}$.

Define J_α by

$$\langle\nabla\rangle^{\delta-\alpha} f = J_{\alpha-\delta} * f, \quad J_\alpha(x) \sim |x|^{\alpha-d}.$$

Need to estimate

$$\mathbb{E}\Big[\big|\langle\nabla\rangle^{\delta-d} : e^{i\beta\Psi_N}(t,x) : \big|^{2p}\Big]$$
$$= e^{p\beta^2\sigma_N(t)} \int_{(\mathbb{T}^2)^p} \mathbb{E}\Big[e^{i\beta\sum_{j=1}^p(\Psi_N(t,y_{2j})-\Psi_N(t,y_{2j-1}))}\Big] \prod_{k=1}^{2p} J_{\alpha-\delta}(x-y_k)\, \mathrm{d}y.$$

Gaussian computation

Expectation is a characteristic function

$$\mathbb{E}\Big[e^{i\beta\sum_{j=1}^{p}(\Psi_N(t,y_{2j})-\Psi_N(t,y_{2j-1}))}\Big]$$
$$= e^{-\frac{\beta^2}{2}\mathbb{E}[|\sum_{j=1}^{p}(\Psi_N(t,y_{2j})-\Psi_N(t,y_{2j-1})|^2]}$$
$$= e^{-\frac{\beta^2}{2}\sum_{j,k=1}^{2p}\epsilon_j\epsilon_k\Gamma_N(t,y_j-y_k)}.$$

Here, Γ_N is the covariance kernel

$$\Gamma_N(t,x-y) := \mathbb{E}[\Psi_N(t,x)\Psi_N(t,y)] \approx -\frac{t}{4\pi}\log(|x-y|+N^{-1}).$$

Obtain

$$\mathbb{E}\Big[|\langle\nabla\rangle^{\delta-d} : e^{i\beta\Psi_N}(t,x) : |^{2p}\Big]$$
$$\lesssim \int_{(\mathbb{T}^2)^p} \Big(\prod_{j,k=1}^{2p} |y_k - y_k|^{-\epsilon_i\epsilon_j \frac{\beta^2 t}{4\pi}} \Big) \prod_{k=1}^{2p} |x - y_k|^{\alpha-d} \mathrm{d}y.$$

Analogy between (static) sine-Gordon and the dimension in other field theories

- Euclidean Φ^3_d with $d = 2 + \frac{\beta^2}{2\pi}$.
- Euclidean Φ^4_d with $d = 2 + \frac{\beta^2}{4\pi}$.
- KPZ with $d = \frac{\beta^2}{4\pi}$.

An inequality

To bound the integrals

$$\int_{(\mathbb{T}^2)^p} \Big(\prod_{j,k=1}^{2p} |y_k - y_k|^{-\epsilon_i \epsilon_j \frac{\beta^2 t}{4\pi}} \Big) \prod_{k=1}^{2p} |x - y_k|^{\alpha - d} \mathrm{d}y$$

use

Lemma

Let $\lambda > 0$ and $p \in \mathbb{N}$. Given $j \in \{1, \dots, 2p\}$, set

$$\epsilon_j = (-1)^j.$$

Let S_p be permutations of $\{1, \dots, p\}$.

For any $\{y_j\}_{j=1,\dots,2p}$ of $2p$ points in $\mathbb{T}^2$ and any $N \in \mathbb{N}$,

$$\prod_{1 \le j < k \le 2p} (|y_j - y_k| + N^{-1})^{\epsilon_j \epsilon_k \lambda} \lesssim \max_{\tau \in S_p} \prod_{1 \le j \le p} (|y_{2j} - y_{2\tau(j)+1}| + N^{-1})^{-\lambda}.$$

An inequality

$$\prod_{1\le j<k\le 2p} (|y_j - y_k| + N^{-1})^{\epsilon_j \epsilon_k \lambda} \lesssim \max_{\tau\in S_p} \prod_{1\le j\le p} (|y_{2j} - y_{2\tau(j)+1}| + N^{-1})^{-\lambda}.$$

- A similar but more general result appears in Hairer-Shen ("dipole computation").
- Froehlich '76 obtains a related estimate by exact identity due to Cauchy

$$\frac{\prod_{1\le i<j\le 2n} |z_i - z_j|^\alpha |w_i - w_j|^\alpha}{\prod_{i,j=1}^{2n} |z_i - w_i|^\alpha} = \left|\det(1/(z_i - w_j))_{1\le i,j\le 2n}\right|.$$

Closing the argument

Residual equation for $v_N = u_N - \Psi_N$

$$v_N(t) = \partial_t S(t)u_0 + S(t)u_1 - \int_0^t S(t-t')\Im(: e^{i\beta\Psi_N} : e^{i\beta v_N}) =: \Gamma(u).$$

Input: $\| : e^{i\beta\Psi_N} : \|_{L^{2p}_\omega L^q_T W_x^{-\alpha,\infty}} \le C$ for $\alpha > \frac{\beta^2 T}{8\pi}$ uniformly in N.

$S(t) = \frac{\sin(t\langle\nabla\rangle)}{\langle\nabla\rangle}$ gains one derivative. Fix T and assume $(u_0, u_1) \in H^s \times H^{1-s}$ for $s = 1-\gamma$ close to 1:

$$\|v_N\|_{H^s} \le \|(u_0,u_1)\|_{H^s\times H^{1-s}} + \left\|\Im(: e^{i\beta\Psi_N} : e^{i\beta v_N})\right\|_{L^1_T H^{s-1}_x}).$$

Closing the argument

$$\|\Im(:e^{i\beta\Psi_N}:e^{i\beta v_N})\|_{L^1_T H^{s-1}_x} \lesssim T^{1/2}\|e^{i\beta v}\|_{L^\infty_T H^{1-s}_x}\|:e^{i\beta\Psi_N}:\|_{L^2_T W_x^{-(1-s),2/(1-s)}}.$$

By "fractional chain rule" (see e.g. Christ and Weinstein, 1991)

$$\|e^{i\beta v_N}\|^2_{H^{1-s}} \lesssim 1+\beta^2\|\langle\nabla\rangle^{1-s}v\|^2_{L^2}. \tag{1}$$

We use a fixed point argument for small T on the probability set

$$\Omega_{T,N}=\{\|:e^{i\beta\Psi_N}:\|_{L^2 W_x^{s-1,\frac{2}{1-s}}}\le 1\}.$$

This works if $1-s$ is small, so that we have the required regularity $-\alpha$ for $:e^{i\beta\Psi_N}:$.

Closing the argument: differences

In estimating the nonlinearity using the fractional chain rule, we did not have to face the nonlineartiy yet. Right side of (1) was of the form "constant + linear in v_N".

For the difference of solutions, we have

$$\begin{aligned}\|\Gamma(v)-\Gamma(w)\|_{L^\infty_T H^s_x} &\leq \|\Im((e^{i\beta v}-e^{i\beta w})e^{i\beta\Psi_N}\|_{L^1_T H^{s-1}} \\ &\lesssim T^{1/2}\|F(v)-F(w)\|_{H^{1-s}},\end{aligned}$$

where

$$F(u)=e^{i\beta u}.$$

Now write

$$F(v)-F(w)=(v-w)\int_0^1 F'(\tau u+(1-\tau)w)\,\mathrm{d}\tau.$$

Estimates for fractional derivatives

To control the product, we use the fractional product rule. Let $s \in [0,1]$. For $r, p_j, q_j \in (1,\infty)$ with $\frac{1}{p_j} + \frac{1}{q_j} = \frac{1}{r}$, $j = 1,2$ then

$$\|fg\|_{W^{s,r}} \lesssim \|f\|_{L^{p_1}} \|g\|_{W^{s,q_1}} + \|f\|_{W^{s,q_2}} \|g\|_{L^{q_2}}.$$

Applying this with $p_1 = \frac{2}{1-s}$, $q_1 = \frac{2}{s}$, $p_2 = \frac{1}{1-s}$:

$$\begin{aligned}
&\|F(v) - F(w)\|_{H^{1-s}} \\
&\lesssim \|v-w\|_{L^{p_1}} \left\| \int_0^1 F'(\tau v + (1-\tau)w)d\tau \right\|_{W^{1-s,q_1}} \\
&+ \|v-w\|_{W^{1-s,p_2}} \left\| \int_0^1 F'(\tau v + (1-\tau)w)d\tau \right\|_{L^{q_2}}.
\end{aligned}$$

We conclude using the Sobolev embeddding.

$$\begin{aligned}
\lesssim &\|v-w\|_{H^s} \left\| \int_0^1 F'(\tau v + (1-\tau)w)d\tau \right\|_{H^s} \\
+ &\|v-w\|_{H^s} \left\| \int_0^1 F'(\tau v + (1-\tau)w)d\tau \right\|_{L^{\frac{2}{s}}}.
\end{aligned}$$

Strichartz estimates

In the argument above, we made an assumption on the regularity s (depending on T, β) of the initial data (u_0, u_1) to close the argument. To access lower regularities, we use mixed space-time *Strichartz spaces*.

Invariant measure

The hyperbolic stochastic sine-Gordon measure

$$\frac{1}{Z}\exp\Big(-\frac{1}{2}\int_{\mathbb{T}^2}|\nabla u|^2\,\mathrm{d}x+\frac{1}{2}\int_{\mathbb{T}^2}u^2\,\mathrm{d}x+\frac{\gamma}{\beta}\int_{\mathbb{T}^2}:\cos(\beta u(x)):\,\mathrm{d}x\Big)$$

is formally invariant

In upcoming work, we use this invariance to construct global solutions to the equation when $\beta^2 < 8\pi$. This type of argument was used by Bourgain (1993).

To construct the measure, we use a recent variational method of Barashkov-Gubinelli (2018).

The end

Thanks for your attention, and Happy Birthday, H.T. Yau!