Fluctuations of the overlap in the 2-spin SSK model at low temperature

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Joint work with P. Sosoe

The Hamiltonian of the Sherrington-Kirkpatrick (SK) model is

$$H_N^{(SK)}(\sigma) = \frac{1}{\sqrt{2N}} \sum_{i \neq j} g_{ij} \sigma_i \sigma_j$$

where

$$\{g_{ij}\}_{i,j=1}^N \sim \text{ iid } \mathcal{N}(0,1)$$

and

$$\{\sigma_i\}_{i=1}^N \in \{-1,+1\}^N$$

- Introduced in 1975 by [SK] as a mean field model of a spin glass with the goal of understanding properties of magnetic alloys with competing ferromagnetic and anti-ferromagnetic interactions
- Scaling is so that the phase transition is at $\beta = 1$.

Parisi formula

Fundamental problem is to calculate the $N \to \infty$ limit $f^{(SK)}(\beta)$ of the free energy

$$F_N^{(SK)}(\beta) := \frac{1}{N} \log Z_N^{(SK)}(\beta),$$

where $Z_N(\beta) = \sum_\sigma \exp\left(-\beta H_N(\sigma)\right)$.

Famously, Parisi (1980) found a variational formula,

$$\lim_{N \to \infty} F_N^{(SK)}(\beta) = \inf_\xi \mathcal{P}_\beta(\xi)$$

where $\mathcal{P}_\beta(\xi)$ is complicated functional, and the infimum is taken over cumulative distribution functions on $[0, 1]$.

The Parisi formula was rigorously proven by Talagrand (2006)

Parisi minimizer and the overlap

Given two replicas $\sigma^{(1)}, \sigma^{(2)}$ (independent samples from the Gibbs measure), the overlap is,

$$R_{12}^{(SK)} := \frac{1}{N}\sigma^{(1)} \cdot \sigma^{(2)}$$

The minimizer in the Parisi formula is interpreted as the limiting distribution of the overlap, and describes the "geometry of the support of the asymptotic Gibbs measure."

In the high temperature phase $\beta < 1$, the Parisi minimizer is the cdf of a trivial random variable and so $R_{12}^{(SK)}$ concentrates around 0.

In the low temperature phase $\beta > 1$, the Parisi minimizer and asymptotic Gibbs measure a complicated ultrametric structure (replica symmetry breaking).

A simpler model is given by the spherical Sherington-Kirkpatrick (SSK) Hamiltonian,

$$H_N(\sigma) = \frac{1}{\sqrt{2N}} \sum_{i \neq j} g_{ij} \sigma_i \sigma_j,$$

where the disorder g_{ij} are iid Gaussians as before.

Replace the ± 1 Ising spins with a continuous phase space:

$$\sigma \in \mathbb{S}^{N-1} := \left\{ \sigma \in \mathbb{R}^N : \sum_i \sigma_i^2 = N \right\}.$$

Note: this is different than replacing each individual spin $\sigma_i \in \pm 1$ with $\sigma_i \in S^1$.

Thermodynamic quantities of interest:

- Partition function is now an integral,

$$Z_N(\beta) := \int_{\mathbb{S}^{N-1}} \mathrm{e}^{-\beta H_N(\sigma)} d\omega_{N-1}(\sigma)$$

  where ω_{N-1} is normalized surface measure on $\mathbb{S}^{N-1}$.
- Free energy has the same form as before

$$F_N(\beta) := \frac{1}{N} \log Z_N(\beta)$$

- Overlap is

$$R_{12} := \frac{1}{N} \sigma^{(1)} \cdot \sigma^{(2)}$$

  where $\sigma^{(i)}$ are independent samples from the Gibbs measure (replicas)

SSK was introduced by Kosterlitz, Thouless and Jones (1976) as a simpler version of the SK model.

[KTJ] calculated the limiting free energy using a contour integral representation and a non-rigorous saddle point analysis:

$$\lim_{N\to\infty} F_N(\beta) = f(\beta) = \begin{cases} \frac{\beta^2}{4} & \beta \leqslant 1 \\ \beta - \frac{\log(\beta)+3/2}{2} & \beta \geqslant 1 \end{cases}.$$

Note that there is a phase transition at $\beta = 1$ where $f(\beta)$ is C^2 but not C^3.

Talagrand (2006) rigorously proved above formula, using similar methods to SK.

Theorem (Baik, Lee, 2015)

Let $F_N(\beta)$ be the SSK free energy and $f(\beta)$ its limiting value as above.

1. In the high temperature regime, $\beta < 1$

$$N(F_N(\beta) - f(\beta)) \to N(m, \alpha)$$

   where $N(m, \alpha)$ is a normal random variable,

2. In the low temperature regime, $\beta > 1$,

$$\frac{2}{\beta - 1} N^{2/3}(F_N(\beta) - f(\beta)) \to \mathrm{TW}_1$$

   where TW_1 is the Tracy-Widom distribution (for the GOE).

- Appearance of random matrix quantities in fluctuations of spin glasses
- High temperature Gaussian fluctuations obtained for SK model by Aizenman, Lebowitz and Ruelle (1987).
- A similar high temperature result appeared earlier in theoretical statistics [Onatski, Moreira and Hallin, 2013].

Recall that the overlap is defined by,

$$R_{12} = \frac{1}{N}\sigma^{(1)} \cdot \sigma^{(2)}$$

where $\sigma^{(i)}$ are two independent samples from the (random) Gibbs measure.

Talagrand and Panchenko proved that R_{12} concentrates about the values $\pm q(\beta)$ where,

$$q(\beta) := \begin{cases} 0 & \beta \leqslant 1 \\ 1 - \frac{1}{\beta} & \beta \geqslant 1 \end{cases}$$

Notation: we will denote expectation wrt the random Gibbs measure by $\langle \cdot \rangle$.

Theorem (Nguyen, Sosoe, 2018)

Let $\langle \cdot \rangle$ be the Gibbs expectation of the SSK model and R_{12} the overlap. In the high temperature phase $\beta < 1$ and for all t,

$$\langle \mathrm{e}^{tR_{12}} \rangle = \mathrm{e}^{t^2} + o(1)$$

with very high probability as $N \to \infty$.

- In particular, R_{12} converges almost surely (with respect to the disorder) to a normal random variable.

- Result holds even for $\beta = \beta_N$ tending to 1 as long as,

$$1 - \beta \geqslant N^{-1/3+\tau}, \qquad \tau > 0$$

- This is expected to be optimal, in that a different distribution should emerge for $1 - \beta \sim N^{-1/3}$
- Annealed result for SK model due to Talagrand

Theorem (L.-Sosoe, 2019)

Let R_{12} be the overlap in the SSK model. In the low temperature phase $\beta > 1$, we have the convergence in distribution of

$$\frac{\beta^2}{2(\beta-1)} \times \lim_{N\to\infty} N^{1/3}\left(\langle R_{12}^2\rangle - q(\beta)^2\right) = \Xi$$

where Ξ is a random variable defined in terms of the Airy$_1$ random point field.

- The presence of the square $\langle R_{12}^2\rangle$ removes the $\pm q(\beta)$ ambiguity.
- Can prove $\langle (R_{12}^2 - q(\beta)^2)^2\rangle \leqslant CN^{-2/3}$ and so a similar result holds for $\langle |R_{12}|\rangle$.
- Presently, only "annealed" result available, but higher moments $\langle (R_{12})^k\rangle$ are in principle accessible.
- Similar results obtained in parallel in forthcoming work of Baik, Le Doussal and Wu by non-rigorous methods (also are investigating the cases of external fields)

Connection to random matrix theory: Note,

$$H_N(\sigma) = \frac{1}{\sqrt{2N}} \sum_{i \neq j} g_{ij} \sigma_i \sigma_j = -\frac{1}{2} \sigma^T M \sigma,$$

where M is a zero-diagonal Gaussian Orthogonal Ensemble matrix:

$$M_{ij} = -\frac{g_{ij} + g_{ji}}{\sqrt{2N}}, \qquad M_{ii} = 0.$$

- **Part 1**: with high probability,

$$\langle R_{12}^2 \rangle - q(\beta)^2 = \frac{2(\beta - 1)}{\beta^2} \left(\frac{1}{N} \sum_{j=2}^{N} \frac{1}{\lambda_j - \lambda_1} + 1 \right) + o(N^{-1/3})$$

  where $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_N$ are the eigenvalues of M.
- **Part 2**: convergence in distribution of

$$\Xi_N := N^{1/3} \left(\frac{1}{N} \sum_{j=2}^{N} \frac{1}{\lambda_j - \lambda_1} + 1 \right) \to \Xi.$$

Due to continuous nature of phase space, observables in the SSK are accessible through contour integral representations:

Lemma

We have,

$$Z_N(\beta) = \int_\Gamma \mathrm{e}^{\frac{N}{2}G(z)}\mathrm{d}z$$

and

$$\langle R_{12}^2 \rangle = \frac{1}{Z_N(\beta)^2} \int_{\Gamma^2} \mathrm{e}^{\frac{N}{2}(G(z)+G(w))} \left(\sum_{i=1}^N \frac{1}{\beta^2 N^2 (z-\lambda_i)(w-\lambda_i)} \right) \mathrm{d}z\mathrm{d}w$$

where $\Gamma = \{\gamma + \mathrm{i}t : t \in \mathbb{R}\}$ and $\gamma > \lambda_1$, and

$$G(z) = \beta z - \frac{1}{N}\sum_{i=1}^N \log(z-\lambda_i).$$

- Such representations used by Kosterlitz-Thouless-Jones, Baik-Lee, Nguyen-Sosoe.

Idea of Lemma: replace the integrals over the $N-1$ sphere:

$$Z_N(\beta) = \int_{\mathbb{S}^{N-1}} \mathrm{e}^{\frac{\beta}{2}\sigma^T M \sigma} \mathrm{d}\omega(\sigma) \to \int_{\mathbb{R}^N} \mathrm{e}^{\frac{\beta}{2}\sigma^T (M-z) \sigma} \mathrm{d}\sigma$$

by an integral over $\mathbb{R}^N$ (and adding a complex convergence factor z) which is a calculable Gaussian integral:

$$\int_{\mathbb{R}^N} \mathrm{e}^{\frac{\beta}{2}\sigma^T (M-z) \sigma} \mathrm{d}\sigma = C_{N,\beta} \prod_j (z-\lambda_j)^{-1/2}$$

On the other hand, after switching to polar coordinates and a change of variable:

$$\int_{\mathbb{R}^N} \mathrm{e}^{\frac{\beta}{2}\sigma^T (M-z) \sigma} \mathrm{d}\sigma = \int_0^\infty \mathrm{e}^{-zr} \mathcal{J}(r) \mathrm{d}r$$

where $\mathcal{J}(r)$ is a spherical integral such that $\mathcal{J}\left(\frac{N\beta}{2}\right) = Z_N(\beta)$.

Apply Laplace inversion formula.

Proof of part 1: Saddle point analysis using contour integral representation

Recall,

$$G(z) = \beta z - \frac{1}{N}\sum_{i=1}^{N} \log(z - \lambda_i), \qquad G'(z) = \beta + \frac{1}{N}\sum_{i=1}^{N} \frac{1}{\lambda_i - z}$$

In the low temperature regime, the saddle γ (i.e., solution to $G'(\gamma) = 0$) is close ($\mathcal{O}(N^{-1})$) to a branch point of the integrand due to the $\log(z - \lambda_1)$ term.

Branch point causes problems in the analysis; use level repulsion of Knowles-Yin to control $\lambda_2 - \lambda_1$, as well as rigidity from Erdős-Schlein-Yau-Yin.

Part 2 of proof: Convergence of $\Xi_N \to \Xi$.

Recall, from part 1:

$$\frac{\beta^2}{2(\beta-1)} N^{1/3}\left(\langle R_{12}^2\rangle - q(\beta)^2\right) = N^{1/3}\left(\frac{1}{N}\sum_{j=2}^{N}\frac{1}{\lambda_j-\lambda_1}+1\right)+o(1) =: \Xi_N + o(1)$$

Two RMT ingredients:

- Scaling limit of the extremal eigenvalues:

$$\left\{N^{2/3}(2-\lambda_i)\right\}_{i=1}^{k} \to \{\chi_i\}_{i=1}^{k}$$

  where $\{\chi_i\}_{i=1}^{\infty}$ is the Airy$_1$ random point field.
- Erdős-Schlein-Yau-Yin rigidity: λ_i concentrates around its classical location γ_i (the N-quantiles of Wigner's semicircle distribution $\rho_{\mathrm{sc}}(E)$).

Basic scheme:

1. Realize that the 1 in Ξ_N is:

$$-1 = \int \frac{1}{E-2}\rho_{\mathrm{sc}}(E)\mathrm{d}E \approx \frac{1}{N}\sum_{j=2}^{\infty}\frac{1}{\gamma_j - \gamma_1}$$

2. Write Ξ_N as,

$$\begin{aligned}
\Xi_N &= \frac{1}{N^{2/3}}\sum_{j=2}^{N}\left(\frac{1}{\lambda_j - \lambda_1} + 1\right) \\
&\approx \frac{1}{N^{2/3}}\sum_{j=2}^{N}\left(\frac{1}{\lambda_j - \lambda_1} - \frac{1}{\gamma_j - \gamma_1}\right) \\
&= \frac{1}{N^{2/3}}\sum_{j=2}^{K}\left(\frac{1}{\lambda_j - \lambda_1} - \frac{1}{\gamma_j - \gamma_1}\right) \\
&\quad + (\text{Error Term})\,.
\end{aligned}$$

For fixed $K > 0$, the first term converges to

$$\lim_{N\to\infty} \frac{1}{N^{2/3}} \sum_{j=2}^{N} \left(\frac{1}{\lambda_j - \lambda_1} - \frac{1}{\gamma_j - \gamma_1} \right) = -\sum_{j=2}^{K} \left(\frac{1}{\chi_j - \chi_1} - \frac{1}{\left(\frac{3\pi j}{2}\right)^{2/3} - \left(\frac{3\pi}{2}\right)^{2/3}} \right)$$

So, define Ξ to be

$$\Xi := -\lim_{K\to\infty} \sum_{j=2}^{K} \left(\frac{1}{\chi_j - \chi_1} - \frac{1}{\left(\frac{3\pi j}{2}\right)^{2/3} - \left(\frac{3\pi}{2}\right)^{2/3}} \right)$$

- But there is an interchange of limits!
- How to deal with the (Error Term)?

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- But there is an interchange of limits!
- How to deal with the (Error Term)?

I am a student of Yau, so try rigidity!

Try to use rigidity: $|\lambda_j - \gamma_j| \leqslant N^{-\frac{2}{3}+\varepsilon} j^{-\frac{1}{3}}$, for any $\varepsilon > 0$.

$$\left| \frac{1}{N^{2/3}} \sum_{j=K+1}^{N} \frac{1}{\lambda_j - \lambda_1} - \frac{1}{\gamma_j - \gamma_1} \right| \leqslant \left| \frac{1}{N^{2/3}} \sum_{j=K+1}^{N} \frac{|\lambda_1 - \gamma_1| + |\lambda_j - \gamma_j|}{(\lambda_j - \lambda_1)(\gamma_j - \gamma_1)} \right|$$

$$\leqslant \frac{N^{\varepsilon}}{N^{2/3}} \left| \frac{1}{N^{2/3}} \sum_{j=K+1}^{N} \frac{1}{(\lambda_j - \lambda_1)(\gamma_j - \gamma_1)} \right|$$

$$\leqslant C N^{\varepsilon} \sum_{j>K} \frac{1}{j^{4/3}} \leqslant C \frac{N^{\varepsilon}}{K^{1/3}}$$

- We lose a polynomial factor - need to take $K \gtrsim N^{3\varepsilon}$.
- Proving convergence of first N^{ε} eigenvalues to Airy_1 seems beyond reach of literature.
- Erdős-Schlein-Yau-Yin rigidity alone is insufficient.

- For the GUE, Gustafsson (2005) proved that

$$\mathrm{Var}(\mathcal{N}(2 - sN^{-2/3})) \leqslant C(1 + |\log(s)|), \tag{1}$$

where $\mathcal{N}(E) = |\{\lambda_i \geqslant E\}|$ is the eigenvalue counting function.

- Eigenvalue rigidity would lose an N^{ε} factor on RHS of (1)
- Can extend to the GOE using a coupling of Forrester and Rains (1999)
- Use duality $\mathcal{N}(E) < j \iff \lambda_j < E$ to find,

$$\mathbb{E}\left|N^{2/3}(\lambda_j - \gamma_j)\right| \leqslant C\frac{|\log(j)|^2 + 1}{j^{1/3}}.$$

No N dependance on RHS!

- Markov's inequality shows that

$$\left|\frac{1}{N^{2/3}}\sum_{j=K+1}^{N}\frac{1}{\lambda_j - \lambda_1} - \frac{1}{\gamma_j - \gamma_1}\right| = o_K(1)$$

with probability $1 - o_K(1)$.

- What about existence of

$$\Xi := -\lim_{K\to\infty}\sum_{j=2}^{K}\left(\frac{1}{\chi_j-\chi_1}-\frac{1}{\left(\frac{3\pi j}{2}\right)^{2/3}-\left(\frac{3\pi}{2}\right)^{2/3}}\right)$$

- Soshnikov (1999) proved for the Airy$_2$ rpf that

$$\mathrm{Var}(\mathcal{N}(E)) \leqslant C(1+|\log(E)|)$$

  where $\mathcal{N}(E)$ is the Airy$_2$ particle counting function.
- Similarly, we use the Forrester-Rains coupling as well as the fact that Airy$_\beta$ are limits of GOE/GUE to extend this to the Airy$_1$ rpf
- Similar arguments imply the a.s. existence of Ξ.

- Interesting to compare the expansion

$$\langle R_{12}^2 \rangle - q(\beta)^2 = \frac{2(\beta-1)}{\beta^2}\left(\frac{1}{N}\sum_{j=2}^{N}\frac{1}{\lambda_j-\lambda_1}+1\right)+o(N^{-1/3})$$

  with a result of Talagrand and Panchenko (2006)
- They observed that $\mathbb{P}\left[\langle R_{12}^2 \rangle \geqslant q^2+\varepsilon\right] \leqslant \mathrm{e}^{-cN}$ for all positive $\varepsilon > 0$, but observed that $\mathbb{P}\left[\langle R_{12}^2 \rangle \leqslant q^2-\varepsilon\right]$ could not be controlled at the level of large deviations.
- Due to having a relatively large probability error, we can not rigorously address this, but:
  - $\Xi_N \leqslant N^{-1/3+\varepsilon}$ with very high probability due to *eigenvalue rigidity*
  - $\frac{1}{N^{2/3}(\lambda_2-\lambda_1)}$ has a (relatively) heavy negative tail due to

$$\mathbb{P}[N^{2/3}(\lambda_1-\lambda_2) \in (s, s+ds)] \sim s ds$$

    for small s.

Looking forward:

- Investigate the case of a magnetic field $H_N(\sigma) + h\sigma \cdot v$, for general v. Different scaling regimes for h (Fyodorov- Le Doussal), and different statistics.
- Find order of fluctuations for $F_N(\beta)$ at $\beta = 1$. For SK (and SSK by same method) Chen and Lam find $\mathrm{O}(\log(N)/N)$. Likely that it is $\mathrm{O}(\sqrt{\log(N)}/N)$.
- "Quenched" result for R_{12}^2 - calculation of higher moments?

Happy Birthday!

Thank you to the organizers for a wonderful conference!