Some problems in hyperbolic hydrodynamic limits: random masses and non-linear wave equation

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Hydrodynamic Scaling Limits

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- Corresponding to different paramenters there are different *partial* equilibriums:
  - *mechanical equilibrium*: constant pressure or tension profiles,
  - *thermal equilibrium*: constant temperature profiles.
- These partial equilibriums may be reached at different time scales: *typically* mechanical equilibrium is reached faster than thermal equilibrium.

Mechanical and Thermal equilibrium

- **Mechanical Equilibrium** is reached in **hyperbolic** time scales (same rescaling of space and time), and is driven by Euler system of equations (for a compressible gas). It involves the ballistic evolution of the long waves (mechanical modes).

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- If thermal conductivity is infinite, **Thermal Equilibrium** is reached in a **super-diffusive** time scales ($\text{time}^\alpha = \text{space}, \alpha < 2$), and typically temperature (or thermal energy) profiles evolve following a *fractional heat equation*.

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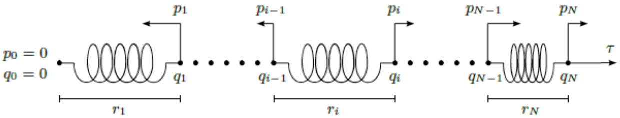
Most of non-equilibrium situation are obtained by

- changing boundary conditions in time
- applying boundary conditions corresponding to different equilibrium states, obtaining dynamics that have *non-equilibrium stationary states* (NESS).

Chain of oscillators



$$
\begin{aligned}
\dot r_x(t) &= p_x(t) - p_{x-1}(t), \qquad && x = 1, \dots, N\\
\dot p_x(t) &= V'(r_{x+1}(t)) - V'(r_x(t)) && x = 1, \dots, N-1\\
\dot p_N(t) &= \tau(t/N) - V'(r_N(t))\\
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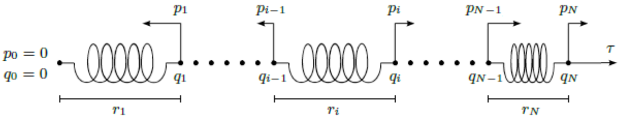

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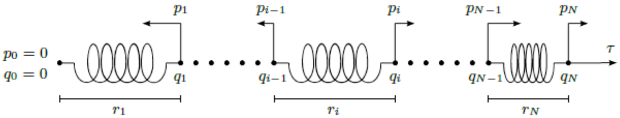

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We are interested in the *macroscopic* evolution of $(r_x(t), p_x(t), \mathcal{E}_x(t))$.

Gibbs measures and Thermodynamic Entropy

For $\tau(t) = \tau$ constant in time, a class of stationary measures is given by the Gibbs measures at temperature β^{-1}, tension τ

$$d\mu_{\beta,\tau,p} = \prod_{x=1}^{N} e^{-\beta(\mathcal{E}_x - \tau r_x) - \mathcal{G}(\beta,\tau)} dp_x dr_x$$

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Thermodynamic entropy is

$$S(u,r) = \inf_{\tau,\beta} \{-\beta\tau r + \beta u - \mathcal{G}(\beta,\tau)\}$$

$$\beta(u,r) = \partial_u S(u,r), \qquad \tau(u,r) = -\beta^{-1}\partial_r S(u,r).$$

Hyperbolic Scaling, Euler equations

3 conserved quantities: we expect the weak convergence to the hyperbolic system of PDE

$$\frac{1}{N}\sum_x G(x/N)\begin{pmatrix} r_x(Nt) \\ p_x(Nt) \\ \mathcal{E}_x(Nt) \end{pmatrix} \underset{N\to\infty}{\longrightarrow} \int_0^1 G(y)\begin{pmatrix} r(y,t) \\ p(y,t) \\ \mathfrak{e}(y,t) \end{pmatrix} dy$$

$$\begin{aligned} \partial_t r(t,y) &= \partial_y p(t,y) \\ \partial_t p(t,y) &= \partial_y \tau[u(t,y), r(t,y)] \\ \partial_t \mathfrak{e}(t,y) &= \partial_y(\tau[u(t,y), r(t,y)] p(t,y)) \end{aligned}$$

where $u = \mathfrak{e} - p^2/2$: internal energy.
and, *for smooth solutions*, the boundary conditions:

$$p(t,0) = 0, \qquad \tau[u(t,1), r(t,1)] = \tau(t)$$

Results with conservative stochastic dynamics

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- Random exchanges of velocities between nearest neighbor particles, conserve energy, momentum and volume, destroying all other (possible) conservation laws. It provides the *right ergodicity* property.
- With such noise in the dynamics, for smooth solutions the HL is proven in:
  - N. Even, S.O., ARMA (2014) (with boundary conditions),
  - SO, SRS Varadhan, HT Yau, CMP (1993) (periodic bc).

Harmonic Oscillators Chain

This is an example of a non-ergodic dynamics:

$$V(r) = r^2/2$$

in fact it is a *completely integrable dynamics*:

$$\dot{q}_x = p_x, \qquad \dot{p}_x = \Delta q_x = q_{x+1} + q_{x-1} - q_x,$$

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Take here $x = 1, \dots, N$,

$$\hat{f}(k) = \sum_x f_x e^{i2\pi kx} \qquad k \in \{0, 1/N, \dots, (N-1)/N\}$$

$\omega(k) = 2|\sin(\pi k)|$ dispersion relation:

$$\mathcal{H} = \sum_x \mathcal{E}_x = \frac{1}{2N} \sum_k \left[\omega(k)^2 |\hat{q}(k)|^2 + |\hat{p}(k)|^2\right]$$

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$$\frac{d}{dt}\hat{\psi}(t,k) = -i\omega(k)\hat{\psi}(t,k) \qquad \hat{\psi}(t,k) = e^{-i\omega(k)t}\hat{\psi}(0,k)$$

Harmonic Oscillators Chain: Quantum Dynamics

$$p_x = -i\partial_{q_x} = -i\left(\partial_{r_{x+1}} - \partial_{r_x}\right)$$

$$\mathcal{E}_x = \frac{1}{2}\left(p_x^2 + r_x^2\right)$$

$$a_k = \frac{1}{\omega(k)}\hat{\psi}(k), \qquad a_k^* = \frac{1}{\omega(k)}\hat{\psi}(k)^*$$

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Heisenber evolution $\frac{d}{dt}A(t) = i[\mathcal{H}, A(t)]$

$$a_k(t) = e^{-i\omega(k)t} a_k, \qquad a_k^*(t) = e^{-i\omega(k)t} a_k^*.$$

Harmonic Chain: Thermal Equilibrium (Classic case)

Consider the chain in *thermal* equilibrium: initial distribution with covariances

$$\langle\ r_x(0); r_{x'}(0)\ \rangle = \langle\ p_x(0); p_{x'}(0)\ \rangle = \beta^{-1}\delta_{x,x'}, \qquad \langle\ q_x; p_{x'}\ \rangle = 0,$$

for some inverse temperature β, while in *mechanical local equilibrium*:

$$\langle\ r_{[Ny]}(0)\ \rangle \longrightarrow r(0,y), \qquad \langle\ p_{[Ny]}(0)\ \rangle \longrightarrow p(0,y).$$

Harmonic Chain: Thermal Equilibrium (classic case)

thermal equilibrium is conserved by the dynamics: for any $t \geq 0$

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Proof.

Thermal equilibrium is Fourier space is:

$$\langle\, \hat{\psi}(k,0)^*; \hat{\psi}(k',0) \,\rangle = 2\beta^{-1}\delta(k-k'), \qquad \langle\, \hat{\psi}(k,0); \hat{\psi}(k',0) \,\rangle = 0.$$

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Consequently

$$\langle\, \hat\psi(k,t)^*; \hat\psi(k',t)\,\rangle = e^{i(\omega(k)-\omega(k'))t}\langle\, \hat\psi(k,0)^*; \hat\psi(k',0)\,\rangle = 2\beta^{-1}\delta(k-k'$$

$$\langle\, \hat\psi(k,t); \hat\psi(k',t)\,\rangle = e^{-i(\omega(k)+\omega(k'))t}\langle\, \hat\psi(k,0); \hat\psi(k',0)\,\rangle = 0.$$

□

Harmonic Chain: Thermal Equilibrium implies Euler Equation limit

$r_{[Ny]}(Nt)$ and $p_{[Ny]}(Nt)$ converge weakly to the solution of the linear wave equation

$$\partial_t \mathbf{r}(y,t) = \partial_y \mathbf{p}(y,t), \qquad \partial_t \mathbf{p}(y,t) = \partial_y \mathbf{r}(y,t).$$

This is the Euler equation for this system since here $\tau(u,r) = r$.

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For the energy, because of the thermal equilibrium, for any $t \geq 0$:

$$\langle\, \mathcal{E}_x(t)\,\rangle = \beta^{-1} + \frac{1}{2}\left(\langle\, p_x(t)\,\rangle^2 + \langle\, r_x(t)\,\rangle^2\right)$$

$$\langle\, \mathcal{E}_{[Ny]}(Nt)\,\rangle \longrightarrow \mathbf{e}(y,t) = \beta^{-1} + \frac{1}{2}\left(\mathbf{p}^2(y,t) + \mathbf{r}^2(y,t)\right),$$

$$\partial_t \mathbf{e}(y,t) = \partial_y\left(\mathbf{p}(y,t)\,\mathbf{r}(y,t)\right).$$

Quantum Harmonic Chain: Thermal Equilibrium

Initial density matrix ρ_β, define

$$\langle A \rangle = \operatorname{tr}\left(A\rho_\beta\right)), \qquad \langle A; B \rangle = \langle AB \rangle - \langle A \rangle\langle B \rangle$$

such that

$$\langle r_x(0); r_{x'}(0) \rangle = \langle p_x(0); p_{x'}(0) \rangle = C_\beta(x-x'), \qquad \langle q_x; p_{x'} \rangle = \frac{i}{2}\delta(x-x')$$

$$C_\beta(x) = \frac{1}{N}\left[\beta^{-1} + \sum_{k\neq 0} e^{2\pi i k x}\left(\frac{\omega_k}{e^{\beta\omega_k} - 1} + \frac{\omega_k}{2}\right)\right] \qquad (1)$$

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$$\langle \mathcal{E}_{[Ny]} \rangle \longrightarrow \mathbf{e}(y) = \bar{C}(\beta) + \frac{1}{2}\left(\mathbf{p}^2(y) + \mathbf{r}^2(y)\right),$$

$$\bar{C}(\beta) = \int_0^1 \omega(k)\left(\frac{1}{e^{\beta\omega(k)}-1} + \frac{1}{2}\right)dk \underset{\beta\to 0}{\sim} \beta^{-1}$$

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Harmonic Chain: Local Thermal Equilibrium is not conserved

The argument fails dramatically if the system is not in thermal equilibrium, even local thermal Gibbs

$$\langle\, r_x(0); r_{x'}(0)\, \rangle = \langle\, p_x(0); p_{x'}(0)\, \rangle = \beta^{-1}\left(\frac{x}{N}\right)\delta_{x,x'}, \quad \langle\, q_x(0); p_{x'}(0)\, \rangle = 0 \tag{2}$$

is not conserved, and correlations between $p_x(t)$ and $r_x(t)$ build up in time.
No autonomous macroscopic equation for the energy!
There are infinite many conservation laws.

Harmonic Chain with Random Masses

The problem with the harmonic chain is that thermal waves of wavenumber k *move* with speed $\omega'(k)$, if they are not uniformed distributed (i.e. the system is not in thermal equilibrium), the temperature profile will not remain constant, as it should be following the Euler equations.

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If the masses are random, the thermal modes remains localized (frozen), by Anderson localization. This allows to close the energy equation, **without local equilibrium**.

Harmonic Chain with Random Masses

(F. Huveneers, C. Bernardin, S.Olla, CMP 2019)

$\{m_x\}$ i.i.d. with absolutely continuous distribution,
$0 < m_- \leq m_x \leq m_+$,
$\overline{m} = \mathbb{E}(m_x)$.

$$m_x \dot{q}_x(t) = p_x(t), \qquad \dot{p}_x(t) = \Delta q_x(t), \qquad x = 1, \dots, N$$

with $q_0 = q_1$ and $q_{N+1} = q_N$ as boundary conditions.

Harmonic Chain with Random Masses: hydrodynamic limit

Almost surely with respect to $\{m_x\}$:

$$< r_{[Ny]}(Nt) >, < p_{[Ny]}(Nt) >, < \mathcal{E}_{[Ny]}(Nt) > \rightharpoonup (\mathbf{r}(y,t), \mathbf{p}(y,t), \mathfrak{e}(y,t))$$

$$\partial_t \mathbf{r}(t,y) = \frac{1}{\overline{m}} \partial_y \mathbf{p}(t,y)$$

$$\partial_t \mathbf{p}(t,y) = \partial_y \mathbf{r}(t,y)$$

$$\partial_t \mathfrak{e}(t,y) = \frac{1}{\overline{m}} \partial_y \left(\mathbf{r}(t,y)\mathbf{p}(t,y)\right)$$

with initial conditions:

$$\mathbf{r}(y,0) = r(y), \qquad \mathbf{p}(y,0) = p(y), \qquad \mathfrak{e}(y,0) = \frac{1}{\beta(y)} + \frac{p^2(y)}{2\overline{m}} + \frac{r^2(y)}{2}.$$

Random Masses: Localization of Thermal Modes

Equation of motion can be written as

$$\ddot{r}_x = -(\nabla^* M^{-1} \nabla r)_x \quad (1 \le x \le N-1), \qquad \ddot{p}_x = \left(\Delta M^{-1} p\right)_x \quad (1 \le x \le N),$$

where $M_{x,x'} = \delta_{x,x'} m_x$.

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$$M^{-1/2}(-\Delta) M^{1/2} \varphi^k = \omega_k^2 \, \varphi^k, \qquad k = 0, \dots, N-1.$$

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$$r(t) = \sum_{k=1}^{N-1} \left(\frac{\langle \nabla \psi^k, r(0) \rangle}{\omega_k} \cos \omega_k t + \langle \psi^k, p(0) \rangle \sin \omega_k t \right) \frac{\nabla \psi^k}{\omega_k},$$

$$p(t) = \sum_{k=0}^{N-1} \left(\langle \psi^k, p(0) \rangle \cos \omega_k t - \frac{\langle \nabla \psi^k, r(0) \rangle}{\omega_k} \sin \omega_k t \right) M \psi^k.$$

Localization of Thermal Modes

Localization length ξ_k diverges with N:

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only the modes $k > \sqrt{N}$ are localized.

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More precisely: for $0 < \alpha < \frac{1}{2}$

$$\mathbb{E}\left(\sum_{k=N^{1-\alpha}}^{N-1} |\psi_x^k \psi_{x'}^k|\right) \leq C e^{-cN^{-2\alpha}|x-x'|}.$$

This estimate is enough to prove that thermal modes remains localized and do not *move* macroscopically.

Random masses: Larger time scales

Assume for simplicity that we are in a *mechanical equilibrium*:

$$\langle\ r_x(0)\ \rangle = 0, \qquad \langle\ p_x(0)\ \rangle = 0,$$

(only thermal energy present)
but not in thermal equilibrium, then, for any $\alpha \geq 1$

$$< \mathcal{E}_{[Ny]}(N^\alpha t) > \underset{N\ to\infty}{\longrightarrow} \mathbf{e}(0,y) = \bar{C}(\beta(y))$$

NO evolution for the temperature profile at any scale!

Random masses: Larger time scales

Assume for simplicity that we are in a *mechanical equilibrium*:

$$\langle\ r_x(0)\ \rangle = 0, \qquad \langle\ p_x(0)\ \rangle = 0,$$

(only thermal energy present)
but not in thermal equilibrium, then, for any $\alpha \geq 1$

$$< \mathcal{E}_{[Ny]}(N^\alpha t) > \underset{N\ to\infty}{\longrightarrow} \mathbf{e}(0, y) = \bar{C}(\beta(y))$$

NO evolution for the temperature profile at any scale!
In particular, for $\alpha = 2$ (diffusive scaling), thermal diffusivity is null.

Anharmonic disordered chain: heat transport

Wojciech De Roeck, Francois Huveneers, S.O., 2019

$$H(q,p) = \sum_{x=1}^{L} \left(\frac{p_x^2}{2} + \omega_x^2 \frac{q_x^2}{2} + g\tau_x \frac{q_x^4}{4} + g_0 \frac{(q_{x+1} - q_x)^2}{2} \right)$$

ω_x i.i.d., $\omega_x^2 \geq \omega_-^2 > 0$
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$\tau_x \in \{0,1\}$, i.i.d., $p = P(\tau_x = 1)$
$j_x = -g_0 p_x (q_{x+1} - q_x)$ energy current,

$$\kappa = \beta^2 \lim_{t\to\infty} \frac{C(t)}{t} \qquad C(t) = \limsup_{L\to\infty} \left\langle \left(\int_0^t ds \frac{1}{\sqrt{L}} \sum_{x=1}^{L-1} j_x(s) \right)^2 \right\rangle_\beta$$

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Kunz-Souillard bound:

$$\mathbb{E}\left(\sum_{k=1}^{L}|\psi_k(x)\psi_k(y)|\right) \;\le\; Ce^{-|x-y|/\xi} \qquad \xi \text{ localization length.}$$

If

$$\gamma \;:=\; \frac{4}{1+(3\xi\log(\frac{1}{1-p}))^{-1}} \;<\; 1,$$

holds, then

$$C(t) = \mathcal{O}((\log t)^5 t^{\gamma}), \qquad \text{i.e.} \qquad \kappa = 0$$

entropy evolution

$$\begin{array}{|c|}\hline \partial_t r = \partial_x p \qquad \partial_t p = \partial_x \tau \qquad \partial_t \mathfrak{e} = \partial_x(\tau p) \\ p(t,0) = 0, \qquad \tau(r(1,t), u(1,t)) = \tau(t) \\ \hline \end{array}$$

$U = \mathfrak{e} - p^2/2$, $\beta = \frac{\partial S}{\partial U}$, $\tau = -\frac{1}{\beta}\frac{\partial S}{\partial r}$

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For smooth solutions:

$$\begin{aligned} \frac{d}{dt} S(u(y,t), r(y,t)) &= \beta\left(\partial_t e - p \partial_t p\right) - \beta \tau \partial_t r \\ &= \beta\left(\partial_x(\tau p) - p \partial_x \tau - \tau \partial_x p\right) = 0 \end{aligned}$$

The evolution is *isoentropic* in the smooth regime.

Shocks, contact discontinuities, weak solutions, entropy solutions

Even starting with initial smooth profiles, hyperbolic non-linear systems develops discontinuities:

- shocks: discontinuities in the tension profile,

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Hydrodynamic limits with shocks

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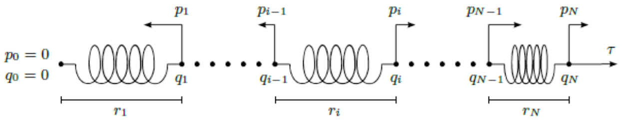
- Well understood for scalar equation (ASEP to Burger, 1992 Rezhakhanlou, ...)

MIcroscopic isothermal dynamics



$$\begin{cases} dr_1 = Np_1 dt + dJ_1^{r,N}(t) \\ dr_i = N(p_i - p_{i-1})dt + dJ_i^{r,N}(t) - dJ_{i-1}^{r,N}(t) \\ dr_N = N(p_N - p_{N-1})dt + dJ_N^{r,N}(t) - dJ_{N-1}^{r,N}(t) \\ dp_1 = N(V'(r_2) - V'(r_1))dt + dJ_0^{p,N}(t) - dJ_1^{p,N}(t) \\ dp_i = N(V'(r_{i+1}) - V'(r_i))dt + dJ_i^{p,N}(t) - dJ_{i-1}^{p,N}(t) \\ dp_N = N(\bar{\tau}(t) - V'(r_N))dt - dJ_{N-1}^{p,N}(t), \end{cases},$$

$$dJ_i^{r,N}(t) = N\sigma_N \left(V'(r_{i+1}) - V'(r_i)\right) dt - \sqrt{2\beta^{-1} N\sigma_N} d\widetilde{w}_i(t)$$

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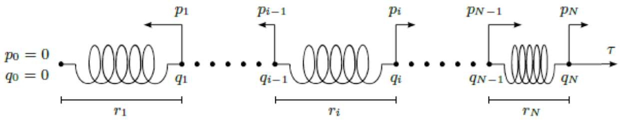

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Hyperbolic Scaling, Euler equations

we expect the weak convergence:

$$\frac{1}{N}\sum_x G(x/N)\begin{pmatrix} r_x(Nt) \\ p_x(Nt)\end{pmatrix} \underset{N\to\infty}{\longrightarrow} \int_0^1 G(y)\begin{pmatrix} r(y,t) \\ p(y,t)\end{pmatrix} dy$$

$$\begin{aligned} r_t &= p_y, & y \in [0,1] \\ p_t &= \tau(r)_y & p(t,0) = 0, \qquad \tau[r(t,1)] = \bar{\tau}(t) \end{aligned}$$

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In the smooth regime of the equations results are obtained even with conservation of energy (Euler equation) with some random exchange of velocities:

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But when shocks appear, we have to consider weak solutions, and from microscopic dynamics we cannot prove any better than L^2 bounds.

Weak solution for the p-system: viscous approximations

$\tau(r) = F'(r)$ and $\tau'(r) = F''(r) > 0.$

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Viscous approximations

$$\begin{aligned} r_t^\delta &= p_y^\delta + \delta r_{yy}^\delta, \\ p_t^\delta &= \tau(r^\delta)_y + \delta p_{yy}^\delta \end{aligned}$$

First question is about the existence of the limit $\delta \to 0$. The main tool is the compensated-compactness (*Tartar, Murat, Ball,* late 70').

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- *S Marchesani, S. Olla, 2018* L^2 solutions with boundaries.

weak solutions of the Cauchy problem with boundary conditions

$$r_t = p_y, \qquad p_t = \tau(r)_y, \qquad y \in [0,1],$$

$$p(t,0) = 0, \qquad \tau[r(t,1)] = \bar{\tau}(t) \qquad (??)$$
$$p(0,y) = p_0(y), \qquad r(0,y) = r_0(y).$$

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$v(t,y) = (r(t,y), p(t,y))$ is a L^2-solution of the Cauchy initial data problem if $t \in [0,T] \to v(t,\cdot)$ is continuous in $L^2(0,1)$, and

$$\int_0^\infty \int_0^1 (\varphi_t r - \varphi_x p)\, dxdt = 0$$

$$\int_0^\infty \int_0^1 (\psi_t p - \psi_x \tau(r))\, dxdt + \int_0^\infty \psi(t,1)\bar{\tau}(t) dt = 0$$

where $\varphi(\cdot,x)$ and $\psi(\cdot,x)$ are compactly supported in $(0,\infty) \times [0,1]$; and $\varphi(t,1) = \psi(t,0) = 0$ for all $t \geq 0$.

Viscosity approximation of the Cauchy Problem with boundaries

$$
\begin{aligned}
r_t^\delta &= p_y^\delta + \delta r_{yy}^\delta, & y \in [0,1], \\
p_t^\delta &= \tau(r^\delta)_y + \delta p_{yy}^\delta
\end{aligned}
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We have two add two boundary conditions, and we choose them to be on Neumann type:

$$\begin{aligned} p^\delta(t,0) &= 0, \qquad \tau(r^\delta(t,1)) = \bar{\tau}(t) \\ p_y^\delta(t,1) &= 0, \qquad r_y^\delta(t,0) = 0 \end{aligned}$$

These *Neumann* bc disappear in the limit, and there is not the problem to compute the boundary layer (that are not even defined for silution in L^2).

Viscosity approximation of the Cauchy Problem with boundaries

Assume some technical conditions on $\tau(r)$:

- $c_1 \leq \tau'(r) \leq c_2$ for some $c_1, c_2 > 0$ and all $r \in \mathbb{R}$;
- $\tau''(r) \neq 0$ for all $r \in \mathbb{R}$;
- $\tau''(r)(\tau'(r))^{-5/4}, \tau'''(r)(\tau'(r))^{-7/4} \in L^2(\mathbb{R})$,
- $\tau''(r)(\tau'(r))^{-3/4}, \tau'''(r)(\tau'(r))^{-2} \in L^\infty(\mathbb{R})$.

Furthermore $\bar{\tau} : \mathbb{R}_+ \to \mathbb{R}$ is smooth and $\bar{\tau}(t) = \tau_1$ for all $t \geq T_\star$.

$$
\begin{aligned}
r_t^\delta &= p_y^\delta + \delta r_{yy}^\delta, \qquad && y \in [0,1], \\
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Viscosity approximation of the Cauchy Problem with boundaries

Under the above technical conditions on $\tau(r)$, the solution of

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converges in $L^p([0,T] \times [0,1])$, $p < 2$, to the L^2 weak solution of Cauchy problem that satisfy the **Clausius inequality**:

$$\mathcal{F}(v(t)) - \mathcal{F}(v(0)) \le W(t), \qquad \forall t \ge 0$$

$$\mathcal{F}(r,p) = \int_0^1 \left(\frac{p(y)^2}{2} + F(r(y)) \right) dy \qquad \text{free energy}$$

$$W(t) = -\int_0^t \int_0^1 \bar\tau'(s) r(s,x) dx ds + \int_0^1 \left(\bar\tau(t) r(t,x) - \bar\tau(0) r_0(x) \right) dx$$

$$= \int_0^t \int_0^1 \bar\tau(s) \partial_s r(s,x) dx ds \qquad \text{work done by } \bar\tau.$$

Clausius inequality (entropy condition)

This is uniformly satisfied by the viscous solution (thanks to the boundary conditions chosen): $\forall t \geq 0$

$$W^\delta(t) = \int_0^t \bar{\tau}(s) dL^\delta(s), \qquad L(s) := \int_0^1 r^\delta(s,x) dx$$

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$$\mathcal{F}(v^\delta(t)) - \mathcal{F}(v(0)) \leq W^\delta(t) - \delta \int_0^t \int_0^1 \left(\tau'(r^\delta)(r_x^\delta)^2 + (p_x^\delta)^2\right) dxds$$
$$\leq W^\delta(t) - \delta(C \wedge 1) \int_0^t \int_0^1 \left((r_x^\delta)^2 + (p_x^\delta)^2\right) dxds$$

since $F''(r) = \tau'(r) \geq C > 0$.

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In this sense any limit point is an *entropy solution*.
Of course it is a very challenging problem to prove uniqueness of these solutions.