Eigenvalues and eigenvectors of critical Erdős-Rényi graphs

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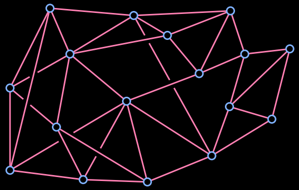
Erdős-Rényi graph and critical regime

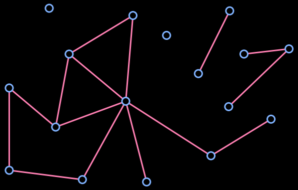
Erdős-Rényi graph $G(N, d/N)$: random graph on N vertices where each edge $\{i, j\}$ is chosen independently with probability d/N.

We consider $N \to \infty$ and $d \equiv d_N$.

Critical regime: $d \approx \log N$, below which degrees do not concentrate.



$d \gg \log N$



$d \ll \log N$

Supercritical $d \gg \log N$: homogeneous.

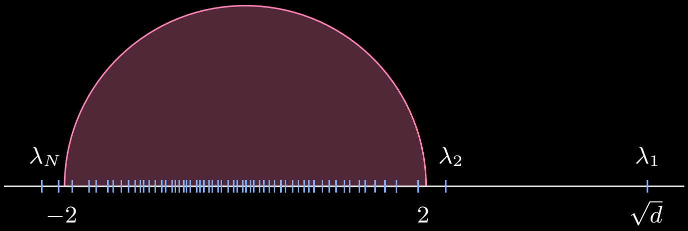
Subcritical $d \ll \log N$: inhomogeneous (hubs, leaves, isolated vertices, ...).

Eigenvalues and eigenvectors

Let $A = (A_{xy}) \in \{0,1\}^{N \times N}$ be the adjacency matrix of $G(N, d/N)$.

Denote by $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_N$ the eigenvalues and $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_N \in \mathbb{S}^{N-1}$ the associated eigenvectors of $d^{-1/2}A$.

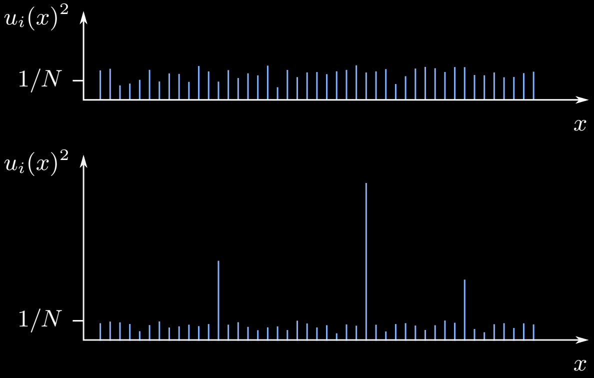
Then [Wigner; 1955] the empirical measure $\frac{1}{N}\sum_i \delta_{\lambda_i}$ converges to the semicircle law on $[-2, 2]$ iff $d \to \infty$.



Key questions in spectral graph theory

(a) Extremal eigenvalues. Convergence, fluctuations.

(b) Eigenvector (de)localization. Delocalization: $\|\mathbf{u}_i\|_\infty^2 \leqslant N^{-1+o(1)}$.



(Very incomplete) summary of previous results

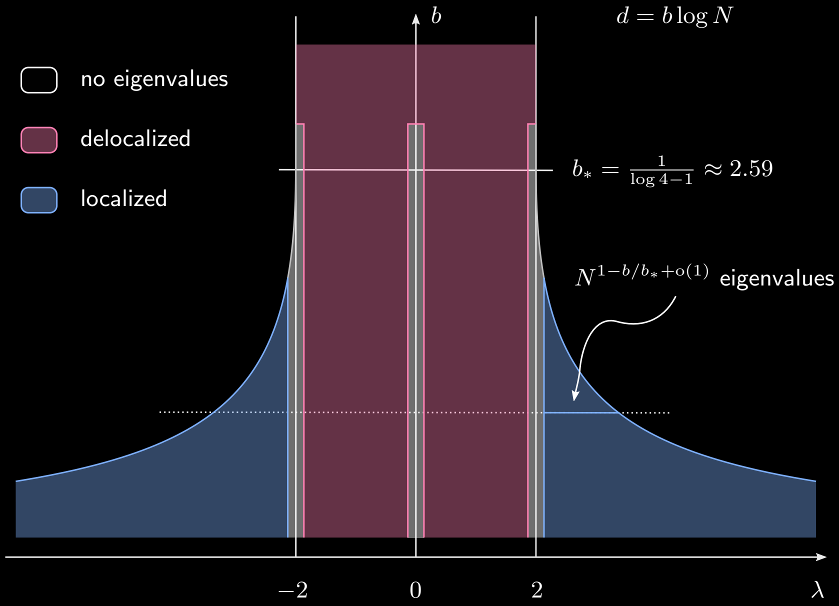
- [Vu; 2007]: If $d \gg (\log N)^4$ then $\lambda_2 = 2 + \mathrm{o}(1)$.
- [Erdős, K, Yau, Yin; 2012]: If $d \gg (\log N)^6$ then delocalization everywhere. If $d \gg N^{2/3}$ then λ_2 has Tracy-Widom fluctuations.
- [Lee, Schnelli; 2016]: If $d \gg N^{1/3}$ then λ_2 has Tracy-Widom fluctuations.
- [Huang, Landon, Yau; 2017]: If $N^{2/9} \ll d \ll N^{1/3}$ then λ_2 has Gaussian fluctuations.
- [Bordenave, Benaych-Georges, K; 2017]: If $d \gg \log N$ then $\lambda_2 = 2 + \mathrm{o}(1)$.
- [Bordenave, Benaych-Georges, K; 2017]: If $d \ll \log N$ then

$$\lambda_2 = (1 + \mathrm{o}(1)) \sqrt{\frac{\max_x \sum_y A_{xy}}{d}} = (1 + \mathrm{o}(1)) \sqrt{\frac{(\log N)/d}{\log((\log N)/d)}} \,.$$

Results: overview

From now on, consider only the (unique) giant component of $G(N, d/N)$.

Phase diagram in the (λ, b)-plane, where λ is an eigenvalue and $d = b \log N$.



b
$d = b \log N$
no eigenvalues
delocalized
localized
$b_* = \frac{1}{\log 4 - 1} \approx 2.59$
$N^{1-b/b_*+o(1)}$ eigenvalues
-2
0
2
λ

Results I: extremal eigenvalues

Define $\alpha_x := \frac{1}{d} \sum_y A_{xy}$ and let $\sigma \in S_N$ satisfy $\alpha_{\sigma(1)} \geqslant \alpha_{\sigma(2)} \geqslant \cdots \geqslant \alpha_{\sigma(N)}$.

Theorem [Alt, Ducatez, K; 2019]. Suppose $(\log N)^{1-c} \leqslant d \leqslant N/2$. Set

$$L := \max\{l \geqslant 1 : \alpha_{\sigma(l)} \geqslant 2 + \mathrm{o}(1)\}\,.$$

Then with very high probability for $1 \leqslant l \leqslant L$ we have

$$|\lambda_{l+1} - \Lambda(\alpha_{\sigma(l)})| \leqslant d^{-c}\,, \qquad \Lambda(\alpha) := \frac{\alpha}{\sqrt{\alpha - 1}}\,, \tag{1}$$

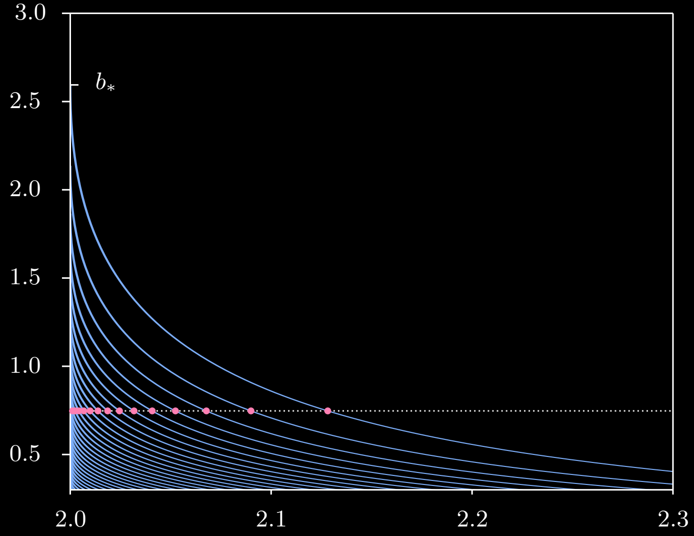
and

$$|\lambda_{L+2} - 2| \leqslant (\log d)^{-c}\,. \tag{2}$$

Remark. Qualitative version of this result for $l = O(1)$ was independently proved by [Tikhomirov, Youssef; 2019].

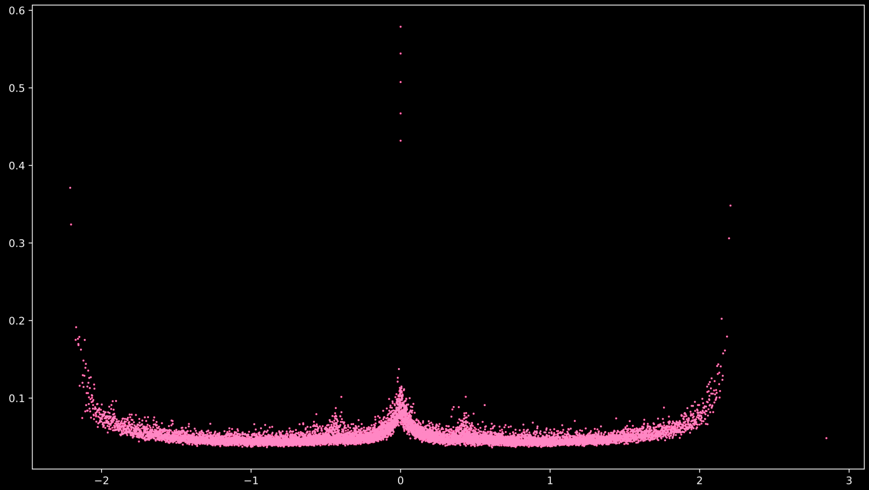
Remark. For the subcritical regime $d \ll \log N$ and $\alpha_{\sigma(l)} \gg 1$, (1) was proved in [Bordenave, Benaych-Georges, K; 2017] using a perturbative argument.

Combine with typical behaviour of degree sequence $\alpha_{\sigma(1)}, \alpha_{\sigma(2)}, \ldots$: (1) occurs if and only if $b > b_* := \frac{1}{\log 4 - 1}$. Graphical analysis of typical behaviour of $\Lambda(\alpha_{\sigma(l)})$ as a function of $b = d/\log N$: (λ, b)-plane for $N = 50$



To the eigenvectors

Simulation: scatter plot of $(\lambda_i, \|\mathbf{u}_i\|_\infty)$. $(N = 10'000,\ b = 0.6)$



Results II: delocalization

Recall: delocalization at λ_i means $\|\mathbf{u}_i\|_\infty^2 \leqslant N^{-1+o(1)}$.

Theorem [He, K, Marcozzi; 2018]. If $d \geqslant C \log N$ then delocalization everywhere.

Theorem [Alt, Ducatez, K; 2019+]. If $d \geqslant C\sqrt{\log N}$ then delocalization in

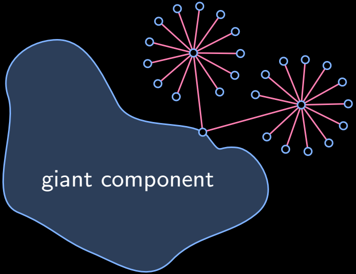
$$\{E \in \mathbb{R} : o(1) \leqslant |E| \leqslant 2 - o(1)\}\,.$$

Remark. The assumptions are optimal in both cases, up to constant C.

Consider two identical stars of central degrees D attached to a common vertex.

This gives rise to a localized eigenvector with eigenvalue $\sqrt{D/d}$.

Such pairs occur up to $D = O(1)$ if $d \leqslant C \log N$ and up to $D = O(d)$ for $d \leqslant C\sqrt{\log N}$.



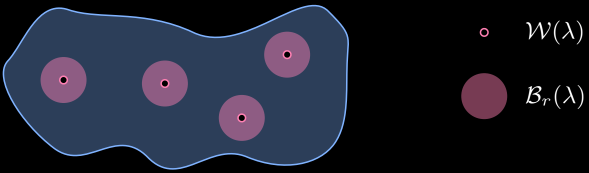
Results III: localization

Theorem [Alt, Ducatez, K; 2019+]. Let $\lambda \geqslant 2 + \mathrm{o}(1)$ be an eigenvalue with eigenvector $\mathbf{u} \in \mathbb{S}^{N-1}$. Define the set of vertices in resonance with λ,

$$\mathcal{W}(\lambda) := \left\{x : \alpha_x \geqslant 2, |\Lambda(\alpha_x) - \lambda| = \mathrm{o}(1)\right\}, \qquad \Lambda(\alpha) := \frac{\alpha}{\sqrt{\alpha - 1}}.$$

For $r \geqslant 1$ define the resonant balls $\mathcal{B}_r(\lambda) := \bigcup_{x \in \mathcal{W}(\lambda)} B_r(x)$. Then for $r \gg 1$, with very high probability,

$$\sum_{x \notin \mathcal{B}_r(\lambda)} u(x)^2 = \mathrm{o}(1), \qquad \sum_{x \in \mathcal{W}(\lambda)} u(x)^2 \geqslant c.$$



Spatial structure of the localized states

For each $x \in \mathcal{W}(\lambda)$ define the spherically symmetric vector

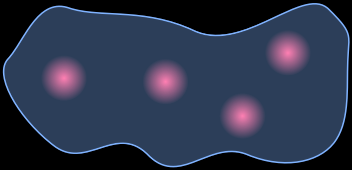
$$\mathbf{v}^{(x)} := \sum_{i=0}^{r} w_i^{(x)} \frac{\mathbf{1}_{S_i(x)}}{\|\mathbf{1}_{S_i(x)}\|},$$

where

$$w_1^{(x)} = \frac{\sqrt{\alpha_x}}{\sqrt{\alpha_x - 1}} w_0^{(x)}, \qquad w_{i+1}^{(x)} = \frac{1}{\sqrt{\alpha_x - 1}} w_i^{(x)} \quad (i \geqslant 1).$$

Let Π denote the orthogonal projection onto $\mathrm{Span}\{\mathbf{v}^{(x)} : x \in \mathcal{W}(\lambda)\}$. Then

$$\|(1 - \Pi)\mathbf{u}\| = \mathrm{o}(1).$$



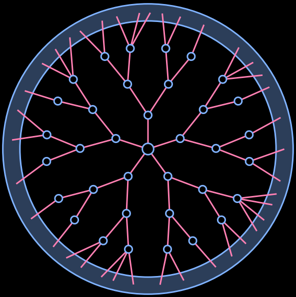
Outline of proof for locations of extremal eigenvalues

Basic observation: The normalized degrees $\alpha_x = |S_1(x)|/d$ do not concentrate. However, if α_x is sufficiently large then there is $r \gg 1$ such that with very high probability:

(a) For each $1 \leqslant i \leqslant r$, the ratio $|S_{i+1}(x)|/|S_i(x)|$ concentrates around d.

(b) The subgraph $G|_{B_r(x)}$ is a tree up to a bounded number of edges.

Consider the toy tree graph $\mathcal{T}$ on N vertices: root x with degree $\alpha_x d$, up to radius r all other vertices have degree $d+1$.



Use the tridiagonal representation of A around x: Let $\mathbf{f}_0, \dots, \mathbf{f}_r$ be the orthonormalization of $\mathbf{1}_x, A\mathbf{1}_x, A^2\mathbf{1}_x, \dots, A^r\mathbf{1}_x$, completed to an orthonormal basis of $\mathbb{R}^N$. Define

$$M^{(x)} := F^* A F\,, \qquad F = [\mathbf{f}_0 \mathbf{f}_1 \cdots \mathbf{f}_{N-1}]\,.$$

Then $M^{(x)}$ is tridiagonal.

For the tree $\mathcal{T}$, the upper $(r+1) \times (r+1)$ block of $M^{(x)}$ is

$$\sqrt{d} \begin{pmatrix} 0 & \sqrt{\alpha_x} & & & & \\ \sqrt{\alpha_x} & 0 & 1 & & & \\ & 1 & 0 & 1 & & \\ & & 1 & 0 & \ddots & \\ & & & \ddots & \ddots & 1 \\ & & & & 1 & 0 \end{pmatrix}.$$

For $\alpha_x \leqslant 2$, spectrum is in $[-2, 2]$. For $\alpha_x > 2$, there are two eigenvalues $\pm\Lambda(\alpha_x)$ outside $[-2, 2]$.

Let $\mathbf{w}$ be the eigenvector corresponding to $\Lambda(\alpha_x)$. A transfer matrix analysis yields

$$w_1^{(x)} = \frac{\sqrt{\alpha_x}}{\sqrt{\alpha_x - 1}} w_0^{(x)}, \qquad w_{i+1}^{(x)} = \frac{1}{\sqrt{\alpha_x - 1}} w_i^{(x)} \quad (1 \leqslant i \leqslant r).$$

Exponential decay for $\alpha_x > 2$.

Back to full graph $G(N, d/N)$: we expect that for $\alpha_x > 2$ the vector

$$\mathbf{v}^{(x)} := \sum_{i=0}^{r} w_i^{(x)} \frac{\mathbf{1}_{S_i(x)}}{\|\mathbf{1}_{S_i(x)}\|},$$

is an approximate eigenvector with eigenvalue near $\Lambda(\alpha_x)$. This is in fact true.

Two key steps in proof:

(L) Every vertex x with $\alpha_x > 2$ gives rise to a unique eigenvalue near $\Lambda(\alpha_x)$.
Lower bound on λ_k.

(U) There are no other eigenvalues in $[2 + \mathrm{o}(1), \infty)$.
Upper bound on λ_k.

For Step (L), we construct a subgraph $G_2 \subset G$ such that

- G_2 is close to G (in some appropriate sense).
- All balls $\{B_r^{G_2}(x) : \alpha_x \geqslant 2\}$ are disjoint.

Then by previous construction all approximate eigenvectors are orthogonal $\Rightarrow$ unique eigenvalues.

For the proof of Step (U), consider for simplicity $H := d^{-1/2}(A - \mathbb{E}A)$, with eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \cdots$. (Going back easy.)

Let $\mathcal{V} := \{x : \alpha_x \geqslant 2\}$. By Step (L), it suffices to prove that $\lambda_{|\mathcal{V}|+1} \leqslant 2 + \mathrm{o}(1)$.

By min-max principle,

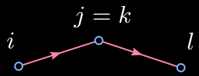
$$\lambda_{|\mathcal{V}|+1} \leqslant \max_{\mathbf{w} \in \mathbb{S}(\mathcal{V})} \langle \mathbf{w}, H\mathbf{w} \rangle, \qquad \mathbb{S}(\mathcal{V}) := \{\mathbf{w} \in \mathbb{S}^{N-1} : w(x) = 0 \, \forall x \in \mathcal{V}\}.$$

Let the maximum be attained at $\tilde{\mathbf{w}}$.

Lemma 1. $H \leqslant I + D + \mathrm{o}(1)$ where $D = \mathrm{diag}(\alpha_1, \ldots, \alpha_N)$.

Proof. Define the nonbacktracking matrix $B = (B_{ef})_{e,f \in [N]^2}$ associated with H through

$$B_{(ij)(kl)} := H_{kl} \mathbf{1}_{j=k} \mathbf{1}_{i \neq l}.$$



Then, by [Bordenave, Benaych-Georges, K; 2017], $\rho(B) = 1 + \mathrm{o}(1)$. Moreover, using an Ihara-Bass-type formula from [Bordenave, Benaych-Georges, K; 2017], we deduce $H \leqslant \rho(B) + D + \mathrm{o}(1)$. □

Using Lemma 1, we deduce that

$$\lambda_{|\mathcal{V}|+1} - \mathrm{o}(1) \leqslant \langle \tilde{\mathbf{w}}, (I + D)\tilde{\mathbf{w}} \rangle \,,$$

where $\tilde{\mathbf{w}}$ is the largest eigenvalue of $H|_{[N]\setminus\mathcal{V}}$. We choose $1 < \tau < 2$ and write the right-hand-side as

$$1 + \sum_{x:\alpha_x < \tau} \alpha_x \tilde{w}(x)^2 + \sum_{x:\tau < \alpha_x \leqslant 2} \alpha_x \tilde{w}(x)^2 \,.$$

Choosing $\tau = 1 + \mathrm{o}(1)$, we conclude that

$$\lambda_{|\mathcal{V}|+1} \leqslant 2 + \mathrm{o}(1) + \sum_{x:\tau < \alpha_x \leqslant 2} \alpha_x \tilde{w}(x)^2 \,.$$

We'll be done if we can prove the following delocalization estimate.

Lemma 2. $\sum_{x:\tau<\alpha_x\leqslant 2} \alpha_x \tilde{w}(x)^2 = \mathrm{o}(1)$ for $\tau = 1 + \mathrm{o}(1)$.

Proof of Lemma 2. As before, we construct a subgraph $G_\tau \subset G$ such that

- G_τ is close to G (in some appropriate sense).
- All balls $\{B_r^{G_\tau}(x) : \alpha_x \geqslant \tau\}$ are disjoint.

Then the main work is to prove that

$$\tilde{w}(x)^2 \leqslant \mathrm{o}(1) \left\| \tilde{\mathbf{w}}|_{B_r^{G_\tau}(x)} \right\|^2 \tag{3}$$

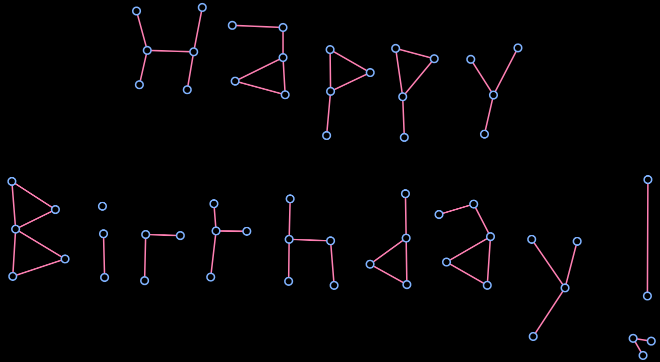
whenever $\alpha_x \geqslant \tau$. We do this using the tridiagonal representation around x.

Using (3) we conclude

$$\sum_{x : \tau < \alpha_x \leqslant 2} \alpha_x \tilde{w}(x)^2 \leqslant 2\,\mathrm{o}(1) \sum_{x : \tau < \alpha_x \leqslant 2} \left\| \tilde{\mathbf{w}}|_{B_r^{G_\tau}(x)} \right\|^2 \leqslant 2\mathrm{o}(1)\,,$$

by disjointness of balls. □

To Yau:



And many more happy mathematical adventures!