SAGE Certificates of Signomial and Polynomial Nonnegativity

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Joint work with Venkat Chandrasekaran and Adam Wierman (Caltech).

Signomials

Signomials are functions of the form

$$\boldsymbol{x} \mapsto \sum_{i=1}^{m} c_i \exp(\boldsymbol{\alpha}_i \cdot \boldsymbol{x})$$

for real scalars c_i, and row vectors $\boldsymbol{\alpha}_i$ in $\mathbb{R}^n$.

Write $f = \mathrm{Sig}(\boldsymbol{\alpha}, \boldsymbol{c})$ for an $m \times n$ matrix $\boldsymbol{\alpha}$, and $\boldsymbol{c}$ in $\mathbb{R}^m$.

Signomials have no concept of degree. We measure a signomial's "complexity" by number of terms needed in the monomial basis

$$\{\boldsymbol{x} \mapsto \exp(\boldsymbol{a} \cdot \boldsymbol{x}) \,:\, \boldsymbol{a} \in \mathbb{R}^n\}.$$

The signomial nonnegativity cone

Caltech

Define the nonnegativity cone for signomials over exponents $\boldsymbol{\alpha}$:

$$C_{\mathrm{NNS}}(\boldsymbol{\alpha}) \doteq \{\, \boldsymbol{c} \,:\, \mathrm{Sig}(\boldsymbol{\alpha}, \boldsymbol{c})(\boldsymbol{x}) \geq 0 \text{ for all } \boldsymbol{x} \text{ in } \mathbb{R}^n \}.$$

These nonnegativity cones exhibit affine-invariance:

$$C_{\mathrm{NNS}}(\boldsymbol{\alpha}) = C_{\mathrm{NNS}}(\boldsymbol{\alpha}\boldsymbol{V}) = C_{\mathrm{NNS}}(\boldsymbol{\alpha} - \mathbf{1}\boldsymbol{u})$$

for all invertible $\boldsymbol{V}$ in $\mathbb{R}^{n \times n}$, and all row vectors $\boldsymbol{u}$ in $\mathbb{R}^n$.

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Checking membership in $C_{\text{NNS}}(\boldsymbol{\alpha})$ ...

- is NP-Hard (for general $\boldsymbol{\alpha}$).
- has applications in engineering design problems.
- is useful for certifying global polynomial nonnegativity.

SAGE is sufficient for nonnegativity

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Definition. A nonnegative signomial with at most one negative coefficient is an "**AM/GM Exponential**," or an "AGE function."

For each k, have cone of coefficients for AM/GM Exponentials

$$C_{\mathrm{AGE}}(\boldsymbol{\alpha}, k) \doteq \{\boldsymbol{c} \;:\; \boldsymbol{c}_{\setminus k} \geq \mathbf{0} \text{ and } \boldsymbol{c} \text{ in } C_{\mathrm{NNS}}(\boldsymbol{\alpha})\}.$$

We take sums of AGE cones to obtain the **SAGE cone**

$$C_{\mathrm{SAGE}}(\boldsymbol{\alpha}) = \sum_{k=1}^{m} C_{\mathrm{AGE}}(\boldsymbol{\alpha}, k).$$

Crucial question: How to represent the AGE cones?

The convex duality behind AGE cones

Caltech

Fix $\boldsymbol{\alpha}$ in $\mathbb{R}^{m \times n}$, and $\boldsymbol{c}$ in $\mathbb{R}^m$ satisfying $\boldsymbol{c}_{\setminus k} \geq \mathbf{0}$.

Does $\boldsymbol{c}$ belong to $C_{\mathrm{NNS}}(\boldsymbol{\alpha})$?

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Does $\boldsymbol{c}$ belong to $C_{\mathrm{NNS}}(\boldsymbol{\alpha})$?

Appeal to affine invariance of $C_{\mathrm{NNS}}(\boldsymbol{\alpha})$, and rearrange terms:

$$\begin{aligned}\mathrm{Sig}(\boldsymbol{\alpha}, \boldsymbol{c})(\boldsymbol{x}) \geq 0 \;\Leftrightarrow\; & \mathrm{Sig}(\boldsymbol{\alpha} - \mathbf{1}\boldsymbol{\alpha}_k, \, \boldsymbol{c})(\boldsymbol{x}) \geq 0 \\ & \mathrm{Sig}(\boldsymbol{\alpha}_{\setminus k} - \mathbf{1}\boldsymbol{\alpha}_k, \, \boldsymbol{c}_{\setminus k})(\boldsymbol{x}) \geq -c_k.\end{aligned}$$

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Appeal to convex duality. The nonnegativity condition

$$\inf_{\boldsymbol{x} \in \mathbb{R}^n} \mathrm{Sig}(\boldsymbol{\alpha}_{\setminus k} - \mathbf{1}\boldsymbol{\alpha}_k, \, \boldsymbol{c}_{\setminus k})(\boldsymbol{x}) \geq -c_k$$

holds **if and only if** there exists $\boldsymbol{\nu}$ in $\mathbb{R}^{m-1}$ satisfying

$$D(\boldsymbol{\nu}, \boldsymbol{c}_{\setminus k}) - \boldsymbol{\nu}^\intercal \mathbf{1} \leq c_k \text{ and } [\boldsymbol{\alpha}_{\setminus k} - \mathbf{1}\boldsymbol{\alpha}_k]\boldsymbol{\nu} = \mathbf{0}.$$

Outline

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1. Discuss selected results for SAGE-signomial certificates.

   M., Chandrasekaran, and Wierman – 2018.

2. Define and prove results for SAGE-polynomial certificates.

   M., Chandrasekaran, and Wierman – 2018.

3. A tiny preview of forthcoming work.

Results for the SAGE signomial cone.

Standard-form SAGE decompositions

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Consider a coefficient vector $\boldsymbol{c} \in \mathbb{R}^m$ satisfying

$$c_1, \ldots, c_\ell < 0 \leq c_{\ell+1}, \ldots c_m,$$

and suppose we want to test if $\boldsymbol{c}$ belongs to $C_{\mathrm{SAGE}}(\boldsymbol{\alpha})$.

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Can show that we only need consider $\boldsymbol{c}^{(k)}$ in $C_{\mathrm{AGE}}(\boldsymbol{\alpha}, k)$.

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Can show that we only need consider $\boldsymbol{c}^{(k)}$ in $C_{\mathrm{AGE}}(\boldsymbol{\alpha}, k)$.

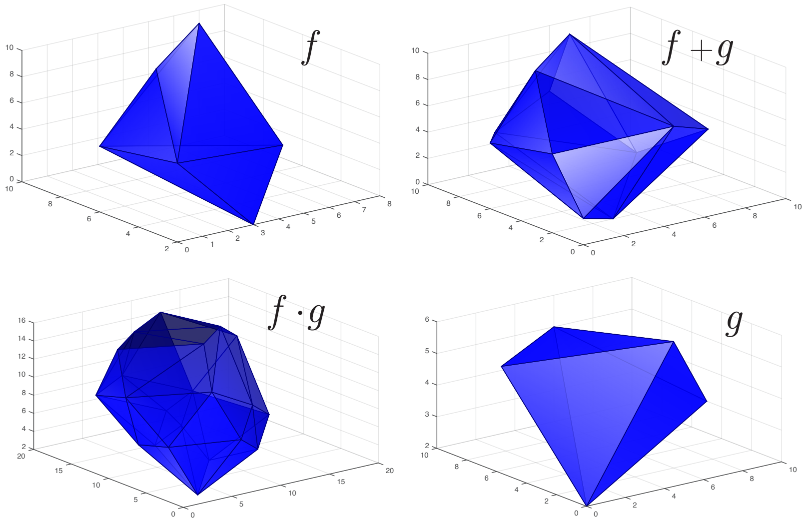
Furthermore, the $\ell \times m$ matrix $\boldsymbol{C}$ with rows "$\boldsymbol{c}^{(k)}$" looks like

$$\boldsymbol{C} = \left[\mathrm{diag}(c_1, \ldots, c_\ell) \,|\, \tilde{\boldsymbol{C}} \right]$$

for some dense, nonnegative $\ell \times (m - \ell)$ matrix $\tilde{\boldsymbol{C}}$.

Think *Newton polytopes*

Caltech

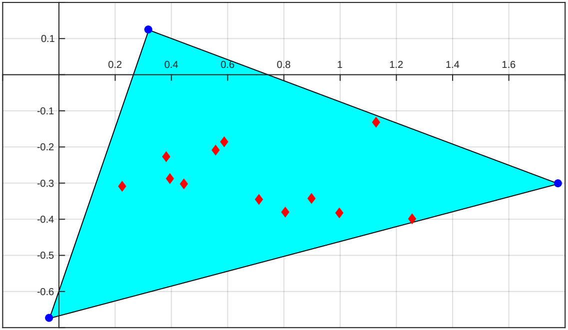


Simplicial sign patterns

Caltech

Theorem (1)

If $\mathrm{Newt}(\boldsymbol{\alpha})$ *is simplicial, and* $c_i \leq 0$ *for all nonextremal* $\boldsymbol{\alpha}_i$*, then* $\boldsymbol{c} \in C_{\mathrm{NNS}}(\boldsymbol{\alpha})$ *if and only if* $\boldsymbol{c} \in C_{\mathrm{SAGE}}(\boldsymbol{\alpha})$*.*



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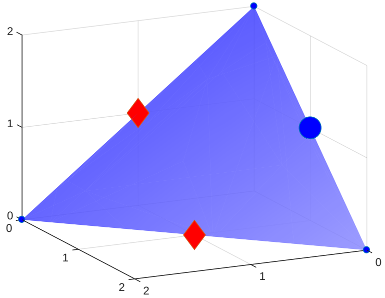


$$f(\boldsymbol{x}) = (e^{x_1} - e^{x_2} - e^{x_3})^2$$

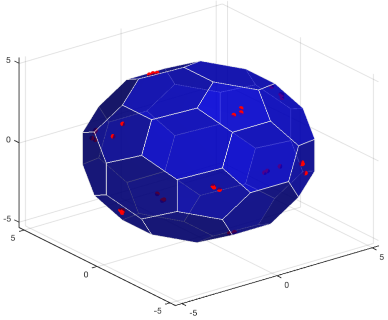
is clearly nonnegative, but

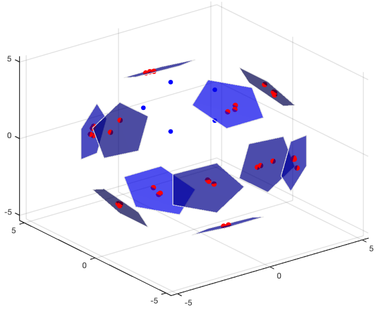
$f - \gamma$ is not SAGE $\forall\ \gamma \in \mathbb{R}$.

Partitioning a Newton polytope

Caltech

We say that $\boldsymbol{\alpha}$ can be **partitioned into ℓ faces** if we can permute its rows so that $\boldsymbol{\alpha} = [\boldsymbol{\alpha}^{(1)}; \ldots ; \boldsymbol{\alpha}^{(\ell)}]$ where $\{\mathrm{Newt}\, \boldsymbol{\alpha}^{(i)}\}_{i=1}^{\ell}$ are mutually disjoint faces of $\mathrm{Newt}(\boldsymbol{\alpha})$.





Partitioning a Newton polytope

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Theorem (2)

If $\{\boldsymbol{\alpha}^{(i)}\}_{i=1}^{\ell}$ *are matrices partitioning* $\boldsymbol{\alpha} = [\boldsymbol{\alpha}^{(1)}; \ldots; \boldsymbol{\alpha}^{(\ell)}]$*, then*

$$C_{\mathrm{NNS}}(\boldsymbol{\alpha}) = \oplus_{i=1}^{\ell} C_{\mathrm{NNS}}(\boldsymbol{\alpha}^{(i)})$$

–and the same is true of $C_{\mathrm{SAGE}}(\boldsymbol{\alpha})$*.*

Sanity checks :

All matrices $\boldsymbol{\alpha}$ admit a trivial partition with $\ell = 1$.

If all $\boldsymbol{\alpha}_i$ are extremal, then $C_{\mathrm{NNS}}(\boldsymbol{\alpha}) = \mathbb{R}_+^m$.

A natural regularity condition: $\boldsymbol{\alpha}$'s *only* partition is trivial.

A Theorem for $C_{\mathrm{SAGE}}(\boldsymbol{\alpha}) = C_{\mathrm{NNS}}(\boldsymbol{\alpha})$

Caltech

Theorem (3)

Suppose $\boldsymbol{\alpha}$ can be partitioned into faces where

1. *each simplicial face has ≤ 2 nonextremal exponents, and*
2. *all other faces contain at most one nonextremal exponent.*

Then $C_{\mathrm{SAGE}}(\boldsymbol{\alpha}) = C_{\mathrm{NNS}}(\boldsymbol{\alpha})$.

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Is the second (nonsimplicial) case too restrictive? Consider

$$\boldsymbol{\alpha}^{\intercal} = \begin{bmatrix} 0 & 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 & 2 \end{bmatrix}.$$

The term $\boldsymbol{\alpha}_6 = [2, 2]$ prevents us from applying Theorem 3.

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Can show $[1.8, -4, 3, -2, 2, 1] \in C_{\mathrm{NNS}}(\boldsymbol{\alpha}) \setminus C_{\mathrm{SAGE}}(\boldsymbol{\alpha})$.

Extreme rays of $C_{\mathrm{SAGE}}(\boldsymbol{\alpha})$

Caltech

A *circuit* is a minimal affinely-dependent pointset of $\mathbb{R}^n$.

We consider circuits "X" that are *simplicial*: $|X \setminus \operatorname{ext} \operatorname{conv} X| = 1$.

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Theorem (4)

If $\boldsymbol{c}$ *generates a nontrivial extreme ray of* $C_{\mathrm{SAGE}}(\boldsymbol{\alpha})$*, then* $\{\boldsymbol{\alpha}_i : c_i \neq 0\}$ *is a circuit.*

The # of circuits induced by $\boldsymbol{\alpha} \in \mathbb{R}^{m \times n}$ can be **exponential in** m.

Possible that **every circuit** supports extreme rays in $C_{\mathrm{SAGE}}(\boldsymbol{\alpha})$.

Yet, we can represent $C_{\mathrm{SAGE}}(\boldsymbol{\alpha})$ with an REP of size $O(m^2)$!

Global Polynomial Nonnnegativity.

Basic definitions

Caltech

Fix $\boldsymbol{\alpha}$ in $\mathbb{N}^{m \times n}$. Write $p = \mathrm{Poly}(\boldsymbol{\alpha}, \boldsymbol{c})$ to mean

$$p(\boldsymbol{x}) = \sum_{i=1}^{m} c_i \boldsymbol{x}^{\boldsymbol{\alpha}_i}, \quad \text{where} \quad \boldsymbol{x}^{\boldsymbol{\alpha}_i} \doteq \prod_{j=1}^{n} x_j^{\alpha_{ij}}.$$

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The matrix $\boldsymbol{\alpha}$ induces a nonnegativity cone

$$C_{\mathrm{NNP}}(\boldsymbol{\alpha}) \doteq \{\, \boldsymbol{c} \,:\, \mathrm{Poly}(\boldsymbol{\alpha}, \boldsymbol{c})(\boldsymbol{x}) \geq 0 \text{ for all } \boldsymbol{x} \text{ in } \mathbb{R}^n \}.$$

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Observe: $\mathrm{Sig}(\boldsymbol{\alpha}, \boldsymbol{c})$ is PSD on $\mathbb{R}^n$ iff $\mathrm{Poly}(\boldsymbol{\alpha}, \boldsymbol{c})$ is PSD on $\mathbb{R}^n_+$.

Thus results for signomials directly extend to even polynomials.

One construction of SAGE polynomials

Caltech

Call $c_i \boldsymbol{x}^{\boldsymbol{\alpha}_i}$ a "monomial square" if $\boldsymbol{\alpha}_i$ is even and $c_i \geq 0$.

p is an "AGE polynomial" – in the monomial basis specified by $\boldsymbol{\alpha}$ – if $p(\boldsymbol{x})$ contains at most one $c_i \boldsymbol{x}^{\boldsymbol{\alpha}_i}$ which is not a monomial square.

In conic form, write

$$C_{\text{AGE}}^{\text{POLY}}(\boldsymbol{\alpha}, k) = \{\, \boldsymbol{c} \,:\, \boldsymbol{c} \in C_{\text{NNP}}(\boldsymbol{\alpha}),\ \boldsymbol{c}_{\setminus k} \geq \mathbf{0},\ \text{and}$$
$$c_i = 0 \text{ for all } i \neq k \text{ with } \boldsymbol{\alpha}_i \notin 2\mathbb{N}^n \}$$

and define

$$C_{\text{SAGE}}^{\text{POLY}}(\boldsymbol{\alpha}) = \sum_{k=1}^{m} C_{\text{AGE}}^{\text{POLY}}(\boldsymbol{\alpha}, k).$$

Another construction, with representation!

Caltech

Define the set of signomial representative coefficient vectors

$$\mathrm{SR}(\boldsymbol{\alpha}, \boldsymbol{c}) = \{\hat{\boldsymbol{c}} : \ \hat{c}_i = c_i \text{ whenever } \boldsymbol{\alpha}_i \text{ is in } 2\mathbb{N}^n, \text{ and}$$
$$\hat{c}_i \leq -|c_i| \text{ whenever } \boldsymbol{\alpha}_i \text{ is not in } 2\mathbb{N}^n\}.$$

If $\hat{\boldsymbol{c}}$ belongs to $\mathrm{SR}(\boldsymbol{\alpha}, \boldsymbol{c})$, then (by a trivial termwise argument)

$$\mathrm{Sig}(\boldsymbol{\alpha}, \hat{\boldsymbol{c}}) \text{ nonnegative} \quad \Rightarrow \quad \mathrm{Poly}(\boldsymbol{\alpha}, \boldsymbol{c}) \text{ nonnegative}.$$

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Theorem (5)

$$C^{\mathrm{POLY}}_{\mathrm{SAGE}}(\boldsymbol{\alpha}) = \{\boldsymbol{c} : \ \mathrm{SR}(\boldsymbol{\alpha}, \boldsymbol{c}) \cap C_{\mathrm{SAGE}}(\boldsymbol{\alpha}) \textit{ is nonempty } \}$$

Theorem 5 can be leveraged to produce many corollaries.

Select corollaries

Caltech

Let p be a polynomial in the monomial basis specified by $\boldsymbol{\alpha}$.

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Let p be a polynomial in the monomial basis specified by $\boldsymbol{\alpha}$.

1. If p is a SAGE polynomial with ℓ terms that are not monomial squares, then it admits a decomposition of ℓ nonnegative polynomials, all of which are supported by exponents $\boldsymbol{\alpha}$.

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3. If p has ≤ 1 extremal term, p is nonnegative iff it is SAGE.

4. The nontrivial extreme rays of $C^{\mathrm{POLY}}_{\mathrm{SAGE}}(\boldsymbol{\alpha})$ are generated by vectors $\boldsymbol{c}$ where $\{\boldsymbol{\alpha}_i \,:\, c_i \neq 0\}$ are simplicial circuits.

Relationship to SONC certificates

Caltech

Corollaries 3 and 4 in the previous slide imply a given polynomial admits a SAGE certificate iff it admits a SONC certificate.

Does this mean SAGE and SONC are the same thing?

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Which is more efficient: SAGE-polynomials, or SONC?

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Can SONC-type analysis still be useful for theoretical purposes?

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Of course!

Polynomial Optimization.

Primal and dual formulations

Caltech

Fix $p = \mathrm{Poly}(\boldsymbol{\alpha}, \boldsymbol{c})$, where exponents $\boldsymbol{\alpha} \in \mathbb{N}^{m \times n}$ have $\boldsymbol{\alpha}_1 = \mathbf{0}$.

The primal SAGE relaxation for $p^\star = \inf_{\boldsymbol{x} \in \mathbb{R}^n} p(\boldsymbol{x})$ is

$$p_{\mathrm{SAGE}} = \sup\{\, \gamma \,:\, \boldsymbol{c} - \gamma \boldsymbol{e}_1 \text{ in } C_{\mathrm{SAGE}}^{\mathrm{POLY}}(\boldsymbol{\alpha})\} \leq p^\star$$

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Applying conic duality, the dual SAGE relaxation is

$$p_{\mathrm{SAGE}} = \inf\{\, \boldsymbol{c}^\intercal \boldsymbol{v} \,:\, \boldsymbol{e}_1^\intercal \boldsymbol{v} = 1,\ \boldsymbol{v} \in C_{\mathrm{SAGE}}^{\mathrm{POLY}}(\boldsymbol{\alpha})^\dagger \}.$$

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If $p_{\mathrm{SAGE}} = p^\star$, how can we recover a minimizer $\boldsymbol{x}^\star \in \mathbb{R}^n$?

Dual solution recovery

Caltech

In terms of standard primitives (LP and REP), can express

$$C^{\mathrm{POLY}}_{\mathrm{SAGE}}(\boldsymbol{\alpha})^{\dagger} = \{\boldsymbol{v} : \text{there exists } \hat{\boldsymbol{v}} \text{ in } C_{\mathrm{SAGE}}(\boldsymbol{\alpha})^{\dagger} \text{ with}$$
$$|\boldsymbol{v}| \leq \hat{\boldsymbol{v}}, \text{ and } v_i = \hat{v}_i \text{ when } \boldsymbol{\alpha}_i \in 2\mathbb{N}^n\}, \text{ and}$$

$$C_{\mathrm{SAGE}}(\boldsymbol{\alpha})^{\dagger} = \{\hat{\boldsymbol{v}} : \text{there exist } \boldsymbol{z}_1, \ldots, \boldsymbol{z}_m \text{ in } \mathbb{R}^n \text{ satisfying}$$
$$\hat{v}_j \log(\hat{\boldsymbol{v}}/\hat{v}_j) \geq [\boldsymbol{\alpha} - \mathbf{1}\boldsymbol{\alpha}_j]\boldsymbol{z}_j \text{ for all } j \text{ in } [m]\}.$$

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Our solution recovery algorithm is simple.

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This procedure comes with guarantees under natural conditions.

Concluding remarks

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Keep an eye on arXiv for

Signomial and Polynomial Optimization via Relative Entropy and Partial Dualization

by Murray, Chandrasekaran, and Wierman.