Algebraic Nevanlinna operator functions and applications to electromagnetics

Christian Engström

Linnaeus University, Sweden

christian.engstrom@lnu.se

Joint work with Heinz Langer, Axel Torshage, Christiane Tretter

October 7, 2019

Outline

EM waves in (non-magnetic) dielectric medium

Maxwell's equations in E

$$\operatorname{curl}\operatorname{curl} E + \frac{\partial^2 D}{\partial t^2} = 0, \quad x = (x_1, x_2, x_3) \in \Omega \subset \mathbb{R}^3$$

with

$$D(x,t) = \left\{ E(x,t) + \int_{-\infty}^{t} K(x, t-\tau) E(x,\tau)\, d\tau \right\}.$$

The Fourier transform $\hat{f}(\omega) = \int e^{i\omega t} f(t) dt$ gives

$$\mathcal{S}(\omega) E = 0, \quad \mathcal{S}(\omega) = \operatorname{curl}\operatorname{curl} - \omega^2 \epsilon(x,\omega),$$

where $\epsilon(x,\omega) = 1 + \hat{K}(x,\omega)$ is the permittivity and $\omega \in \mathcal{D} \subset \mathbb{C}$.

What do we want to know?

Properties of the spectrum: $\sigma(\mathcal{S}) = \{\omega \in \mathcal{D} \, : \, 0 \in \sigma(\mathcal{S}(\omega))\}$

Resolvent estimates: Behaviour of $\|\mathcal{S}^{-1}(\omega)\|$

Properties of the evolution Maxwell equations:

$$\operatorname{curl}\operatorname{curl} E + \frac{\partial^2}{\partial t^2}\left\{ E(x,t) + \int_{-\infty}^{t} K(x, t-\tau) E(x,\tau)\, d\tau \right\} = 0$$

$+$ boundary and initial conditions.

Drude-Lorentz = damped harmonic oscillator

- d - damping
- $\sqrt{c}$ - resonant frequency of undamped oscillator
- $\sqrt{b}$ - plasma frequency

$\theta := \sqrt{c - \frac{d^2}{4}} \neq 0$ (under/over - damping):

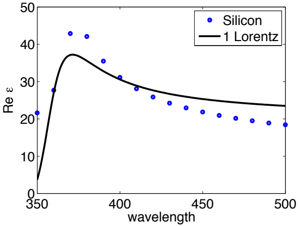
$$K(t) = \frac{b}{\theta} e^{-td/2} \sin(\theta t).$$

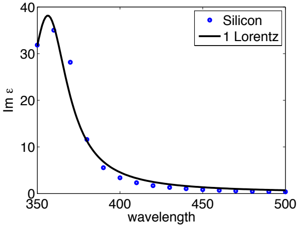
Assume $\theta := 0$ (critical damping):

$$K(t) = bte^{-td/2}$$

- $\Omega = \Omega_1 \cup \Omega_2$
- $\epsilon(x, \omega) = 1 + \hat{K}(x, \omega) := \chi_{\Omega_1}(x) + \epsilon_2(\omega)\chi_{\Omega_2}(x)$

Analytic properties of $\mathcal{S}$?





$$\epsilon_2(\omega) = 1 + \sum_{\ell} \hat{K}_\ell(\omega) = 1 + \sum_{\ell=1}^{L} \frac{b_\ell}{c_\ell - i d_\ell \omega - \omega^2}, \; b_\ell > 0, c_\ell \geqslant 0, d_\ell \geqslant 0.$$

✓ $\omega \mapsto \omega\epsilon(\omega)$ maps $\mathbb{C}^+$ on $\bar{\mathbb{C}}^+$,

- But is the operator function

$$\mathcal{S}(\omega) = \operatorname{curl}\operatorname{curl} - \omega^2 \epsilon(x, \omega)$$

Nevanlinna (after change of variables)?

Consider $\mathcal{S}$ with the multi-pole Drude-Lorentz model:

$$\mathcal{S}(\omega) = A_0 - \omega^2 - \omega^2 \sum_{\ell=1}^{L} \frac{M_\ell}{c_\ell - d_\ell \omega - \omega^2},$$

with $A_0 = \operatorname{curl}\operatorname{curl}$, and $M_\ell = b_\ell \chi_{\Omega_2}$.

Set $\omega = -\sqrt{\lambda}$. Then $-\mathcal{S}(\lambda) : L^2(\Omega)^3 \to L^2(\Omega)^3$ with

$$\mathcal{S}(\lambda) = A_0 - \lambda - \lambda \sum_{\ell=1}^{L} \frac{M_\ell}{c_\ell + i d_\ell \sqrt{\lambda} - \lambda},$$

is Nevanlinna if

- A_0 is self-adjoint & $M_\ell \geqslant 0$
- $d_\ell = 0$ or $c_\ell \leqslant d_\ell^2/4$ for all $\ell = 1, 2, \dots, L$

Consider $\mathcal{S}$ with the multi-pole Drude-Lorentz model:

$$\mathcal{S}(\omega) = A_0 - \omega^2 - \omega^2 \sum_{\ell=1}^{L} \frac{M_\ell}{c_\ell - d_\ell \omega - \omega^2},$$

with $A_0 = \operatorname{curl}\operatorname{curl}$, and $M_\ell = b_\ell \chi_{\Omega_2}$.

Set $\omega = -\sqrt{\lambda}$. Then $-\mathcal{S}(\lambda) : L^2(\Omega)^3 \to L^2(\Omega)^3$ with

$$\mathcal{S}(\lambda) = A_0 - \lambda - \lambda \sum_{\ell=1}^{L} \frac{M_\ell}{c_\ell + i d_\ell \sqrt{\lambda} - \lambda},$$

is Nevanlinna if

- A_0 is self-adjoint & $M_\ell \geqslant 0$
- $d_\ell = 0$ or $c_\ell \leqslant d_\ell^2/4$ for all $\ell = 1, 2, \ldots, L$

Case $d_\ell = 0$

Polynomial long division gives

$$\lambda\epsilon(x,\lambda) = \lambda - \sum_{\ell=1}^{L} b_\ell \chi_{\Omega_2}(x) + \sum_{\ell=1}^{L} \frac{c_\ell b_\ell}{c_\ell - \lambda}\chi_{\Omega_2}(x)$$

Set

- $A = A_0 + \sum_{\ell=1}^{L} b_\ell \chi_{\Omega_2}$
- $B_\ell^* = \sqrt{c_\ell b_\ell}\chi_{\Omega_2}$, where $\chi_{\Omega_2} : L^2(\Omega)^3 \to \widehat{\mathcal{H}}_2$, $\widehat{\mathcal{H}}_2 = \operatorname{ran} \chi_{\Omega_2}$.

Then

$$\mathcal{S}(\lambda) = A - \lambda - \sum_{\ell=1}^{L} \frac{B_\ell B_\ell^*}{c_\ell - \lambda},$$

Equivalent block operator matrix

$$\mathcal{S}(\lambda) = A - \lambda - \sum_{\ell=1}^{L} \frac{B_\ell B_\ell^*}{c_\ell - \lambda}, \quad \operatorname{dom} \mathcal{S}(\lambda) = \operatorname{dom} \mathrm{A}, \quad \lambda \in \mathbb{C} \backslash \{\mathrm{c_1}, \mathrm{c_2}, \ldots, \mathrm{c_L}\},$$

where $B_\ell : \widehat{\mathcal{H}}_2 \to L^2(\Omega)^3$, $\ell = 1, 2, \ldots, L$.

- $\widetilde{\mathcal{H}} = L^2(\Omega)^3 \oplus \widehat{\mathcal{H}}$, $\widehat{\mathcal{H}} = \widehat{\mathcal{H}}_2 \oplus \cdots \oplus \widehat{\mathcal{H}}_2$

$\mathcal{S}$ is the Schur complement of $\mathcal{A} : \widetilde{\mathcal{H}} \to \widetilde{\mathcal{H}}$,

$$\mathcal{A} = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix} = \begin{pmatrix} A & B_1 & B_2 & \cdots & B_L \\ B_1^* & c_1 & 0 & \cdots & 0 \\ B_2^* & 0 & c_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B_L^* & 0 & 0 & \cdots & c_L \end{pmatrix}, \quad \operatorname{dom} \mathcal{A} = \operatorname{dom} A \oplus \widehat{\mathcal{H}}.$$

Classical min-max principle for self-adjoint operators

Assumptions

- A has discrete spectrum, (e.g. $E = (0, 0, u(x_1, x_2))$ in electromagnetics)
- A is self-adjoint and bounded from below
- B_ℓ, $\ell = 1, 2, \dots, L$ are bounded

Then

1. $\sigma_{\mathrm{ess}}(\mathcal{A}) = \{c_1, c_2, \dots, c_L\}$(Adamjan, Atkinson, H. Langer, Mennicken, Shkalikov)
2. $\mathcal{A}$ is self-adjoint and bounded from below

From the min-max principle (Rayleigh-Ritz, Courant-Fischer) follows

$$\lambda_n = \min_{\substack{\mathcal{L} \subset \mathrm{dom}\, \mathcal{A} \\ \dim \mathcal{L} = \mathrm{n}}} \max_{\substack{u \in \mathcal{L} \\ u \neq 0}} p(u), \;\; p(u) := \frac{(\mathcal{A}u, u)}{||u||^2}$$

where $((\mathcal{A} - \lambda)u, u) = 0$ has solution $p(u)$ and $\lambda_n \to \min \sigma_{\mathrm{ess}}(\mathcal{A}) = c_1$.

Classical min-max principle for self-adjoint operators

Assumptions

- A has discrete spectrum, (e.g. $E = (0, 0, u(x_1, x_2))$ in electromagnetics)
- A is self-adjoint and bounded from below
- B_ℓ, $\ell = 1, 2, \ldots, L$ are bounded

Then

1. $\sigma_{\mathrm{ess}}(\mathcal{A}) = \{c_1, c_2, \ldots, c_L\}$(Adamjan, Atkinson, H. Langer, Mennicken, Shkalikov)
2. $\mathcal{A}$ is self-adjoint and bounded from below

From the min-max principle (Rayleigh-Ritz, Courant-Fischer) follows

$$\lambda_n = \min_{\substack{\mathcal{L}\subset \mathrm{dom}\,\mathcal{A} \\ \dim \mathcal{L} = \mathrm{n}}} \max_{\substack{u\in\mathcal{L} \\ u\neq 0}} p(u), \;\; p(u) := \frac{(\mathcal{A}u, u)}{||u||^2}$$

where $((\mathcal{A} - \lambda)u, u) = 0$ has solution $p(u)$ and $\lambda_n \to \min \sigma_{\mathrm{ess}}(\mathcal{A}) = c_1$.

Variational principles in $(c_\ell, c_{\ell+1})$?

- $(\mathcal{S}(\lambda)u, u) = 0$ has solution $p_{\ell+1}(u)$ in $(c_\ell, c_{\ell+1})$

From the Nevanlinna property follows

$$\frac{d}{d\lambda}(\mathcal{S}(\lambda)u, u) = -\|u\|^2 - \sum_{\ell=1}^{L} \frac{\|B_\ell^*\|^2}{(c_\ell - \lambda)^2} \leqslant -\|u\|^2, \quad u \in \operatorname{dom}\mathcal{S}, \mathrm{u} \neq 0$$

Morover, $\mathcal{S}(\lambda) = \operatorname{dom}\mathrm{A}$ independent of λ.

- These properties (and some additional) imply variational principles (M. Langer/Eschwé (2004))

Simplified result for one rational term

Assume $A \geqslant c_1$. Then the eigenvalues of $\mathcal{A}$ (and $\mathcal{S}$) are

$$\lambda_{1,n} = \min_{\substack{\mathcal{L}\subset \mathrm{dom\,A} \\ \dim \mathcal{L}=\mathrm{n}}} \max_{\substack{u\in\mathcal{L} \\ u\neq 0}} p_1(u), \quad \lambda_{1,n} \to c_1$$

$$\lambda_{2,n} = \min_{\substack{\mathcal{L}\subset \mathrm{dom\,A} \\ \dim \mathcal{L}=\mathrm{n}}} \max_{\substack{u\in\mathcal{L} \\ u\neq 0}} p_2(u), \quad \lambda_{2,n} \to \infty,$$

where

$$p_{1,2}(u) := \frac{1}{2}\left(\frac{(Au,u)}{\|u\|^2} + c_1\right) \mp \sqrt{\frac{1}{4}\left(\frac{(Au,u)}{\|u\|^2} - c_1\right)^2 + \frac{\|B_1^* u\|^2}{\|u\|^2}}.$$

Note that $p_{1,2}(u)$ are the solutions of $(P(\lambda)u,u) = 0$, where

$$P(\lambda) := (c_1 - \lambda)\mathcal{S}(\lambda) = \lambda^2 - \lambda(A + c_1) - B_1 B_1^*$$

Main results in E./Langer/Tretter (2017)

✓ gaps in the spectrum to the right of c_ℓ, $\ell = 1, \ldots, L$

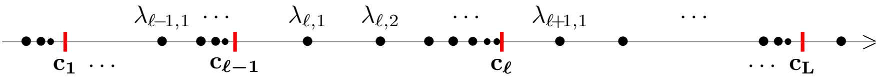
✓ c_ℓ is an accumulation point of eigenvalues of $\mathcal{A}$ from the left

✓ min-max characterisation of the eigenvalues:

$$\lambda_{\ell,n} = \min_{\substack{\mathcal{L}\subset \mathrm{dom}\,\mathrm{A} \\ \dim \mathcal{L} = \mathrm{n}+\kappa_\ell}} \max_{\substack{u\in\mathcal{L} \\ u\neq 0}} p_\ell(u)$$

where κ_ℓ is the number of negative eigenvalues of $\mathcal{S}(\eta_\ell^+)$.

- No index shift (i.e. $\kappa_\ell = 0$) if $A > c_L$.



Extensions to $d_\ell > 0$ (joint work with Axel Torshage)

What can we say about the spectrum of

$$\mathcal{S}(\omega) = A_0 - \omega^2 - \omega^2 \sum_{\ell=1}^{L} \frac{M_\ell}{c_\ell - d_\ell\omega - \omega^2},$$

when $d_\ell > 0$ for some ℓ?

- The tools used when $d_\ell = 0$ can not be applied
- We need different tools and will use theory of bounded operator polynomials (Keldysh, Krein, Langer, Markus, Matsaev, Russu,...)

Theory of polynomial operator functions

The theory is difficult to use since

- We need good knowledge of the numerical range

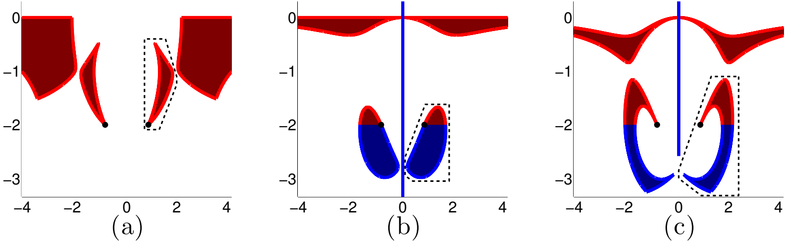
$$W(\mathcal{S}) = \{\omega \in \mathcal{D} :\ \exists u \in \mathrm{dom}\,(\mathrm{A})\backslash\{0\}, \|\mathrm{u}\| = 1, \text{so that}\,(\mathcal{S}(\omega)\mathrm{u},\mathrm{u}) = 0\}$$

- We can only show accumulation of eigenvalues in bounded components of the numerical range

- Reformulate the problem as an operator polynomial P with bounded operator coefficients (of a special form)
- Show that it exists operator polynomials R and Q such that

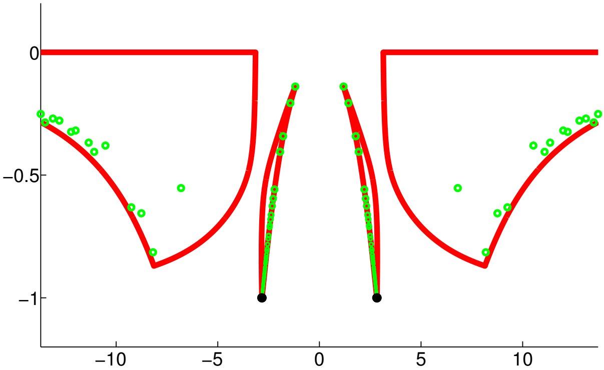
$$P(\omega) = R(\omega)Q(\omega), \sigma(R) = \Gamma \cap \sigma(P), \sigma(Q) \subset \mathbb{C}\setminus\bar{\Gamma},$$

where $\Gamma \subset \mathbb{C}$ is bounded.



Γ is the dotted line ($\mathcal{S}$ has one rational term)

Application to lossy photonic crystal



Poles at $\pm\sqrt{8} - i$ for $\mathcal{S}$ with one rational term

- We can prove accumulation of eigenvalues to the poles
- Solid lines bound the spectrum
- The circles are numerically computed eigenvalues (p-FEM)

Where are we going now?

Other equations

- Full Maxwell's equations with double negative and lossy materials
- Wave equations with viscoelastic materials (Bolzmann integral)
- Scattering resonances (nonlinearity in the DtN-map)

Evolution problems

- Get to know the resolvent → get to know the semigroup

Is $\|\mathcal{S}^{-1}(\lambda)\|$ a Mouse or an Elephant (or a Duck)?

References

C. Tretter. Spectral theory of block operator matrices and applications. Imperial College Press, 2008

A. S. Markus. Introduction to the spectral theory of polynomial operator pencils. AMS, 1988

F. V. Atkinson, H. Langer, R. Mennicken, A. A. Shkalikov. The essential spectrum of some matrix operators. Math. Nachr., 167: 5-20, 1994.

D. Eschwé, M. Langer. Variational principles for eigenvalues of self-adjoint operator functions. IEOT, 49 (3): 287-321, 2004.

C. Engström, A. Torshage, *Spectral properties of conservative, dispersive, and absorptive photonic crystals*, GAMM - Mitteilungen, 41:e201800009, 2018

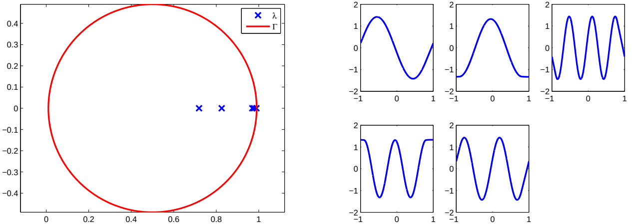
C. Engström, A. Torshage, *Accumulation of complex eigenvalues of a class of analytic operator functions*, J. Funct. Anal., 275 (2), 442-477, 2018

C. Engström, A. Torshage, *Enclosure of the Numerical Range of a Class of Non-selfadjoint Rational Operator Functions*. IEOT, **88** (2), 151–184, 2017

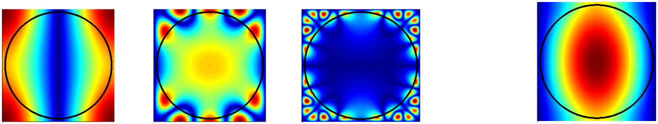
C. Engström, H. Langer, C. Tretter, *Rational eigenvalue problems and applications to photonic crystals*. J. Math. Anal. Appl., **445**(1), 240-279, 2017

Properties of the eigenvectors

Problem in 1D (E./Grubišić, 2015):

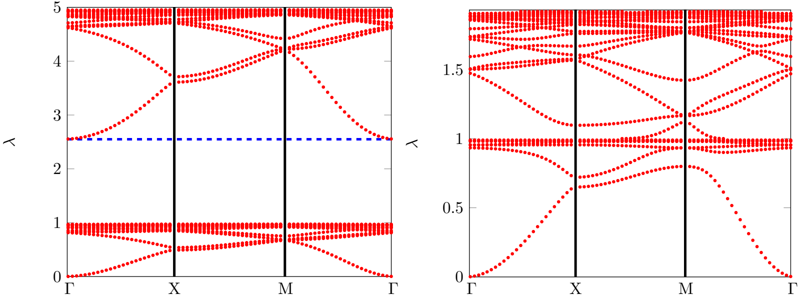


Problem in 2D:



- We can prove that the eigenvectors behave as in the 2D example if the eigenvalues do not accumulate too quickly
- This accumulation rate depends on the geometry!

In general no gap for all k



- We can in some cases guarantee a band gap by using verified eigenvalue enclosures to show that $A_k > c_2$
  Hoang/Plum/Wieners (2009)
- In general no accumulation for fixed k, but no gap for all k

min-max principle for the rational function (main results)

- Define $p_\ell(u) \in [c_{\ell-1}, c_\ell]$ for $u \in \mathrm{dom}\,(\mathrm{A}) = \mathrm{dom}\,(\mathcal{S}(\lambda))$ by

$$p_\ell(u) := \begin{cases} \lambda_\ell(u) & \text{if } \ (\mathcal{S}(\lambda_\ell(u))u, u) = 0 \ \text{ for } \ \lambda_\ell(u) \in (c_{\ell-1}, c_\ell), \\ c_{\ell-1} & \text{if } \ (\mathcal{S}(\lambda)u, u) < 0 \ \text{ for all } \ \lambda \in (c_{\ell-1}, c_\ell), \\ c_\ell & \text{if } \ (\mathcal{S}(\lambda)u, u) > 0 \ \text{ for all } \ \lambda \in (c_{\ell-1}, c_\ell), \end{cases}$$

✓ The spectrum of $\mathcal{S}$ consists of $L+1$ eigenvalue sequences $(\lambda_{\ell,j})_{j=1}^{n_\ell} \subset (c_{\ell-1}, c_\ell)$, $n_\ell \in \mathbb{N}_0 \cup \{\infty\}$, which may be characterized as

$$\lambda_{\ell,n} = \min_{\substack{\mathcal{L}\subset \mathrm{dom}\,(\mathrm{A}) \\ \dim \mathcal{L} = \mathrm{n}+\kappa_\ell}} \ \max_{\substack{u\in\mathcal{L} \\ u\neq 0}} p_\ell(u)$$