# Self-Adjoint Boundary Conditions for Singular Sturm–Liouville Operators and the computation of $m$-functions for Bessel, Legendre, and Laguerre operators

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**Herglotz-Nevanlinna Theory Applied to Passive, Causal and Active Systems**

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# Motivation

Try to extend the classical boundary values

$$g(a) = -W(u_a(\lambda_0, \cdot), g)(a) = \lim_{x \downarrow a} \frac{g(x)}{\widehat{u}_a(\lambda_0, x)}, \qquad (*)$$

$$g^{[1]}(a) = (pg')(a) = W(\widehat{u}_a(\lambda_0, \cdot), g)(a) = \lim_{x \downarrow a} \frac{g(x) - g(a)\widehat{u}_a(\lambda_0, x)}{u_a(\lambda_0, x)}, \qquad (**)$$

for **regular Sturm–Liouville operators** on $[a, b] \subset \mathbb{R}$ associated with differential expressions of the type

$$\tau = r(x)^{-1}[-(d/dx)p(x)(d/dx) + q(x)] \text{ for a.e. } x \in [a, b] \subset \mathbb{R},$$

to the case where $\tau$ is singular on $(a, b) \subseteq \mathbb{R}$ and the associated **minimal operator $T_{min}$** is **bounded from below**.

Here $u_a(\lambda_0, \cdot)$ and $\widehat{u}_a(\lambda_0, \cdot)$ denote appropriately normalized **principal** and **nonprincipal** solutions of $\tau u = \lambda_0 u$ for appropriate $\lambda_0 \in \mathbb{R}$, respectively.

While the l.h.s. in $(*)$, $(**)$ above will cease to be meaningful in the singular case, it will be shown that the r.h.s. remains valid!

# Some Literature

**Based on:**

**F.G., L. Littlejohn, and R. Nichols,** *A note on self-adjoint boundary conditions for singular Sturm–Liouville operators*, in preparation.

**Also relies on:**

**H.-D. Niessen and A. Zettl,** *Singular Sturm–Liouville problems: the Friedrichs extension and comparison of eigenvalues*, Proc. London Math. Soc. (3) **64**, 545–578 (1992).

# All Self-Adjoint B.C.'s in the Regular Case

All **separated** and **coupled** boundary conditions together describe **all self-adjoint** extensions of $T_{min}$:

**Theorem.**

Assume that $\tau$ is **regular** on $[a,b]$. Then the following items $(i)$–$(iii)$ hold:

$(i)$ **All self-adjoint** extensions $T_{\alpha,\beta}$ of $T_{min}$ with **separated** boundary conditions are of the form

$$\begin{aligned} &T_{\alpha,\beta} f = \tau f, \quad \alpha, \beta \in [0,\pi), \\ &f \in \mathrm{dom}(T_{\alpha,\beta}) = \big\{ g \in \mathrm{dom}(T_{max}) \,\big|\, g(a)\cos(\alpha) + g^{[1]}(a)\sin(\alpha) = 0; \\ &\qquad\qquad\qquad\qquad\qquad\qquad\quad g(b)\cos(\beta) + g^{[1]}(b)\sin(\beta) = 0 \big\}. \end{aligned}$$

Special cases: $\alpha = 0$, $g(a) = 0$ is called the **Dirichlet** boundary condition at $a$; $\alpha = \frac{\pi}{2}$, $g^{[1]}(a) = 0$ is called the **Neumann** boundary condition at $a$ (analogous facts hold at the endpoint $b$).

**Note.** Here $g^{[1]}(a) = \lim_{x \downarrow a} p(x) g'(x)$, $\;g^{[1]}(b) = \lim_{x \uparrow b} p(x) g'(x)$, denote the first quasi-derivatives of $g$ at $x = a$, resp., at $x = b$.

# All Self-Adjoint B.C.'s in the Regular Case (contd.)

**Theorem (contd.).**

$(ii)$ **All self-adjoint** extensions $T_{\varphi,R}$ of $T_{min}$ with **coupled** boundary conditions are of the type

$$T_{\varphi,R} f = \tau f,$$

$$f \in \operatorname{dom}(T_{\varphi,R}) = \left\{ g \in \operatorname{dom}(T_{max}) \,\middle|\, \begin{pmatrix} g(b) \\ g^{[1]}(b) \end{pmatrix} = e^{i\varphi} R \begin{pmatrix} g(a) \\ g^{[1]}(a) \end{pmatrix} \right\},$$

where $\varphi \in [0, 2\pi)$, and $R$ is a real $2 \times 2$ matrix with $\det(R) = 1$ (i.e., $R \in SL(2, \mathbb{R})$).

Special cases: $\varphi = 0$, $R = I_2$, $g(b) = g(a)$, $g^{[1]}(b) = g^{[1]}(a)$ are called **periodic** boundary conditions; similarly, $\varphi = \pi$, $R = I_2$, $g(b) = -g(a)$, $g^{[1]}(b) = -g^{[1]}(a)$ are called **antiperiodic** boundary conditions.

$(iii)$ **Every self-adjoint** extension of $T_{min}$ is either of type $(i)$ (i.e., **separated**) or of type $(ii)$ (i.e., **coupled**).

This completely characterizes the regular case (standard textbook literature).

# The Singular Case. Basics

**Hypothesis.**

Let $(a,b) \subseteq \mathbb{R}$ and suppose that $p, q, r$ are (Lebesgue ) measurable functions on $(a,b)$ such that the following items $(i)$–$(iii)$ hold:

$(i)$ $r > 0$ a.e. on $(a,b)$, $r \in L^1_{loc}((a,b); dx)$.

$(ii)$ $p > 0$ a.e. on $(a,b)$, $1/p \in L^1_{loc}((a,b); dx)$.

$(iii)$ $q$ is real-valued a.e. on $(a,b)$, $q \in L^1_{loc}((a,b); dx)$.

**Definition.**

The **maximal** operator $T_{max}$ in $L^2((a,b); rdx)$ associated with $\tau$ is defined by

$$\begin{aligned} &T_{max} f = \tau f, \\ &f \in \operatorname{dom}(T_{max}) = \big\{ g \in L^2((a,b); rdx) \,\big|\, g, g^{[1]} \in AC_{loc}((a,b)); \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad \tau g \in L^2((a,b); rdx) \big\}. \end{aligned}$$

# The Singular Case. Basics (cont.)

## Definition (contd.).

The **minimal** operator $T_{min,0}$ in $L^2((a,b);rdx)$ associated with $\tau$ is defined by

$$
\begin{aligned}
&T_{min,0}f = \tau f,\\
&f \in \mathrm{dom}(T_{min,0}) = \big\{g \in L^2((a,b);rdx) \,\big|\, g, g^{[1]} \in AC_{loc}((a,b));\\
&\qquad\qquad \mathrm{supp}\,(g) \subset (a,b) \text{ is compact; } \tau g \in L^2((a,b);rdx)\big\}.
\end{aligned}
$$

One can prove that $T_{min,0}$ is closable and then defines $T_{min}$ as the closure of $T_{min,0}$, $T_{min,0} = \overline{T_{min,0}}$.

# The Singular Case. Basics (cont.)

**Theorem (Weyl's Alternative).**

The following **alternative** holds:

($i$) For every $z \in \mathbb{C}$, all solutions $u$ of $(\tau - z)u = 0$ are in $L^2((a,b); rdx)$ near $b$ (resp., near $a$).

($ii$) For every $z \in \mathbb{C}$, there exists at least one solution $u$ of $(\tau - z)u = 0$ which is not in $L^2((a,b); rdx)$ near $b$ (resp., near $a$). In this case, for each $z \in \mathbb{C}\backslash\mathbb{R}$, there exists precisely one solution $u_b$ (resp., $u_a$) of $(\tau - z)u = 0$ (up to constant multiples) which lies in $L^2((a,b); rdx)$ near $b$ (resp., near $a$).

This yields the **limit circle/limit point** classification of $\tau$ at an interval endpoint:

**Definition (Limit Circle/Limit Point).**

In case ($i$) in the Theorem, $\tau$ is said to be in the **limit circle case** at $b$ (resp., at $a$). (Frequently, $\tau$ is then called **quasi-regular** at $b$ (resp., $a$).)

In case ($ii$) in the Theorem, $\tau$ is said to be in the **limit point case** at $b$ (resp., at $a$).

If $\tau$ is in the **limit circle case** at $a$ and $b$ then $\tau$ is called **quasi-regular** on $(a,b)$.

# All Self-Adjoint B.C.'s in the Singular Case

**Theorem.**

Assume that $\tau$ is in the **limit circle case** at $a$ and $b$ (i.e., $\tau$ is quasi-regular on $(a,b)$). In addition, assume that $v_j \in \operatorname{dom}(T_{max})$, $j = 1, 2$, satisfy

$$W(\overline{v_1}, v_2)(a) = W(\overline{v_1}, v_2)(b) = 1, \quad W(\overline{v_j}, v_j)(a) = W(\overline{v_j}, v_j)(b) = 0, \ j = 1, 2.$$

(E.g., real-valued sols. $v_j$, $j = 1, 2$, of $(\tau - \lambda)u = 0$ with $\lambda \in \mathbb{R}$, s.t. $W(v_1, v_2)$ $= 1$.) For $g \in \operatorname{dom}(T_{max})$ we introduce the **generalized boundary values**

$$\begin{aligned} \widetilde{g}_1(a) &= -W(v_2, g)(a), \quad & \widetilde{g}_1(b) &= -W(v_2, g)(b), \\ \widetilde{g}_2(a) &= W(v_1, g)(a), \quad & \widetilde{g}_2(b) &= W(v_1, g)(b). \end{aligned}$$

Then the following items $(i)$–$(iii)$ hold:

$(i)$ **All self-adjoint** extensions $T_{\alpha,\beta}$ of $T_{min}$ with **separated** b.c.'s are of the form

$$\begin{aligned} &T_{\alpha,\beta} f = \tau f, \quad \alpha, \beta \in [0, \pi), \\ &f \in \operatorname{dom}(T_{\alpha,\beta}) = \big\{ g \in \operatorname{dom}(T_{max}) \, \big| \, \widetilde{g}_1(a)\cos(\alpha) + \widetilde{g}_2(a)\sin(\alpha) = 0; \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad \widetilde{g}_1(b)\cos(\beta) + \widetilde{g}_2(b) sin(\beta) = 0 \big\}. \end{aligned}$$

# All Self-Adjoint B.C.'s in the Singular Case (contd.)

**Theorem (contd.).**

$(ii)$ **All self-adjoint** extensions $T_{\varphi,R}$ of $T_{min}$ with **coupled** boundary conditions are of the type

$$T_{\varphi,R} f = \tau f,$$
$$f \in \operatorname{dom}(T_{\varphi,R}) = \left\{ g \in \operatorname{dom}(T_{max}) \,\middle|\, \begin{pmatrix} \widetilde{g}_1(b) \\ \widetilde{g}_2(b) \end{pmatrix} = e^{i\varphi} R \begin{pmatrix} \widetilde{g}_1(a) \\ \widetilde{g}_2(a) \end{pmatrix} \right\},$$

where $\varphi \in [0, 2\pi)$, and $R$ is a real $2 \times 2$ matrix with $\det(R) = 1$ (i.e., $R \in SL(2, \mathbb{R})$).

$(iii)$ **Every self-adjoint** extension of $T_{min}$ is either of type $(i)$ (i.e., **separated** ) or of type $(ii)$ (i.e., **coupled** ).

# $T_{min}$ Bounded from Below. Basics

**Definition.**

$(i)$ Fix $c \in (a, b)$ and $\lambda \in \mathbb{R}$. Then $\tau - \lambda$ is called **nonoscillatory** at $a$ (resp., $b$), if every real-valued solution $u(\lambda, \cdot\,)$ of $\tau u = \lambda u$ has finitely many zeros in $(a, c)$ (resp., $(c, b)$). Otherwise, $\tau - \lambda$ is called **oscillatory** at $a$ (resp., $b$).

$(ii)$ Let $\lambda_0 \in \mathbb{R}$. Then $T_{min}$ is called **bounded from below** by $\lambda_0$, and one writes $T_{min} \geq \lambda_0 I$, if

$$(u, [T_{min} - \lambda_0 I]u)_{L^2((a,b);rdx)} \geq 0, \quad u \in \mathrm{dom}(T_{min}).$$

The following is a key result.

**Theorem.**

The following items $(i)$–$(ii)$ are equivalent:

$(i)$ $T_{min}$ (and hence any symmetric extension of $T_{min}$) is **bounded from below**.

$(ii)$ There exists a $\nu_0 \in \mathbb{R}$ such that for all $\lambda < \nu_0$, $\tau - \lambda$ is **nonoscillatory** at $a$ and $b$.

# $T_{min}$ Bounded from Below. Basics (contd.)

**Definition.**

Suppose that $T_{min}$ is **bounded from below**, and let $\lambda \in \mathbb{R}$.

($i$) Then $u_a(\lambda, \cdot)$ (resp., $u_b(\lambda, \cdot)$) is called a **principal** (or **minimal**) solution of $\tau u = \lambda u$ at $a$ (resp., $b$) if $u_a(\lambda, \cdot)$ and $u_b(\lambda, \cdot)$ are minimal solutions of $\tau u = \lambda u$ in the sense that

$$u(\lambda, x)^{-1} u_a(\lambda, x) = o(1) \text{ as } x \downarrow a,$$
$$u(\lambda, x)^{-1} u_b(\lambda, x) = o(1) \text{ as } x \uparrow b,$$

for any other solution $u(\lambda, \cdot)$ of $\tau u = \lambda u$ (which is nonvanishing near $a$, resp., $b$) with $W(u_a(\lambda, \cdot), u(\lambda, \cdot)) \neq 0$, respectively, $W(u_b(\lambda, \cdot), u(\lambda, \cdot)) \neq 0$.

($ii$) A real-valued solution $\widehat{u}_a(\lambda, \cdot)$ (resp., $\widehat{u}_b(\lambda, \cdot)$) of $\tau u = \lambda u$ linearly independent of $u_a(\lambda, \cdot)$ (resp., $u_b(\lambda, \cdot)$) is called **nonprincipal** at $a$ (resp., $b$).

# Boundary Values if $T_{min}$ is Bounded from Below

**Theorem.**

Assume that $\tau$ is in the **limit circle case** at $a$ and $b$ (i.e., $\tau$ is quasi-regular on $(a,b)$). In addition, assume that $T_{min} \geq \lambda_0 I$ for some $\lambda_0 \in \mathbb{R}$, and denote by $u_a(\lambda_0,\,\cdot\,)$ and $\widehat{u}_a(\lambda_0,\,\cdot\,)$ (resp., $u_b(\lambda_0,\,\cdot\,)$ and $\widehat{u}_b(\lambda_0,\,\cdot\,)$) principal and nonprincipal solutions of $\tau u = \lambda_0 u$ at $a$ (resp., $b$), satisfying (a normalization)

$$W(\widehat{u}_a(\lambda_0,\,\cdot\,), u_a(\lambda_0,\,\cdot\,)) = W(\widehat{u}_b(\lambda_0,\,\cdot\,), u_b(\lambda_0,\,\cdot\,)) = 1.$$

Introduce $v_j \in \operatorname{dom}(T_{max})$, $j = 1, 2$, via

$$v_1(x) = \begin{cases} \widehat{u}_a(\lambda_0, x), & \text{for } x \text{ near a,} \\ \widehat{u}_b(\lambda_0, x), & \text{for } x \text{ near b,} \end{cases} \quad v_2(x) = \begin{cases} u_a(\lambda_0, x), & \text{for } x \text{ near a,} \\ u_b(\lambda_0, x), & \text{for } x \text{ near b,} \end{cases}$$

# Boundary Values if $T_{min}$ is Bounded from Below

**Theorem (contd.).**

Then one obtains for all $g \in \operatorname{dom}(T_{max})$,

$$\begin{aligned}
\widetilde{g}(a) &= -W(v_2, g)(a) = \widetilde{g}_1(a) = -W(u_a(\lambda_0, \cdot\,), g)(a) \\
&= \lim_{x \downarrow a} \frac{g(x)}{\widehat{u}_a(\lambda_0, x)}, \qquad (*) \\
\widetilde{g}(b) &= -W(v_2, g)(b) = \widetilde{g}_1(b) = -W(u_b(\lambda_0, \cdot\,), g)(b) \\
&= \lim_{x \uparrow b} \frac{g(x)}{\widehat{u}_b(\lambda_0, x)}, \qquad (**) \\
\widetilde{g}\,'(a) &= W(v_1, g)(a) = \widetilde{g}_2(a) = W(\widehat{u}_a(\lambda_0, \cdot\,), g)(a) \\
&= \lim_{x \downarrow a} \frac{g(x) - \widetilde{g}(a)\widehat{u}_a(\lambda_0, x)}{u_a(\lambda_0, x)}, \qquad (***) \\
\widetilde{g}\,'(b) &= W(v_1, g)(b) = \widetilde{g}_2(b) = W(\widehat{u}_b(\lambda_0, \cdot\,), g)(b) \\
&= \lim_{x \uparrow b} \frac{g(x) - \widetilde{g}(b)\widehat{u}_b(\lambda_0, x)}{u_b(\lambda_0, x)}. \qquad (****)
\end{aligned}$$

In particular, the limits on the right-hand sides in $(*)$ – $(****)$ exist.

# The Friedrichs Extension

The Friedrichs extension is characterized in the expected manner:

**Theorem (Niessen and Zettl 1992).**

Assume that $\tau$ is in the **limit circle case** at $a$ and $b$ (i.e., $\tau$ is quasi-regular on $(a,b)$). In addition, assume that $T_{min} \geq \lambda_0 I$ for some $\lambda_0 \in \mathbb{R}$. Then the **Friedrichs extension** $T_F$ of $T_{min}$ is characterized by

$$T_F f = \tau f, \quad f \in \operatorname{dom}(T_F) = \big\{ g \in \operatorname{dom}(T_{max}) \,\big|\, \widetilde{g}(a) = \widetilde{g}(b) = 0 \big\}.$$

We recall,

$$\widetilde{g}(a) = \lim_{x \downarrow a} \frac{g(x)}{\widehat{u}_a(\lambda_0, x)}, \quad \widetilde{g}(b) = \lim_{x \uparrow b} \frac{g(x)}{\widehat{u}_b(\lambda_0, x)}.$$

One can now express Weyl–Titchmarsh $m$-functions directly in terms of the boundary values $\widetilde{g}$, $\widetilde{g}'$, but this needs a few preparations:

# Basics of $m$-function Theory

In the singular Sturm–Liouville operator case this is a bit more involved!
First, one needs a (rather benign) spectral hypothesis:

## Spectral Hypothesis

In addition to the standard assumptions on $p, q, r$, suppose that for some (and hence for all ) $c \in (a, b)$, the self-adjoint operator $T_{\alpha_0,0,a,c}$ in $L^2((a, c); rdx)$, associated with $\tau|_{(a,c)}$ and a **Dirichlet boundary condition** at $c$ (i.e., $g(c) = 0$, $g \in \text{dom}(T_{max,a,c})$, the maximal operator associated with $\tau|_{(a,c)}$ in $L^2((a, c); rdx)$), has **purely discrete spectrum**.

This Hypothesis is equivalent to the existence of an **entire** solution $\phi_{\alpha_0}(z, \cdot\,)$ of $\tau u = zu$, $z \in \mathbb{C}$, that is **real-valued** for $z \in \mathbb{R}$, and lies in $\text{dom}(T_{\alpha_0,\beta_0})$ near the point $a$. In particular, $\phi_{\alpha_0}(z, \cdot\,)$ satisfies the **boundary condition** indexed by $\alpha_0$ at the left endpoint $a$ if $\tau$ is in the **limit circle case** at $a$, and $\phi_{\alpha_0}(z, \cdot\,) \in L^2((a, c); rdx)$ if $\tau$ is in the **limit point case** at $a$. In addition, a second, linearly independent **entire** solution $\theta_{\alpha_0}(z, \cdot\,)$ of $\tau u = zu$ exists, with $\theta_{\alpha_0}(z, \cdot\,)$ **real-valued** for $z \in \mathbb{R}$, satisfying (the normalization)

$$W(\theta_{\alpha_0}(z, \cdot\,), \phi_{\alpha_0}(z, \cdot\,)) = 1, \quad z \in \mathbb{C}.$$

# Basics of $m$-function Theory (contd.)

We note that $\phi_{\alpha_0}(z,\cdot)$ is unique up to a nonvanishing entire factor (real on the real line) with respect to $z \in \mathbb{C}$. Hence, we may normalize $\phi_{\alpha_0}(z,\cdot)$ such that

$$\widetilde{\phi}_{\alpha_0}(z,a) = -\sin(\alpha_0), \quad \widetilde{\phi}'_{\alpha_0}(z,a) = \cos(\alpha_0), \quad z \in \mathbb{C},$$

and thus,

$$\widetilde{\theta}_{\alpha_0}(z,a) = \cos(\alpha_0), \qquad \widetilde{\theta}'_{\alpha_0}(z,a) = \sin(\alpha_0), \quad z \in \mathbb{C},$$

Given (for $z \in \mathbb{C}\backslash\mathbb{R}$),

$$\psi_{\beta_0,+}(z,\cdot) = \theta_0(z,\cdot) + m_{0,\beta_0}(z)\phi_0(z,\cdot) \begin{cases} \text{satisfies the } \textbf{b.c.} \text{ at } x=b \\ \qquad\qquad \text{if } \tau \text{ is } \textbf{l.c.c.} \text{ at } b, \\ \in L^2((a,b);r(x)dx) \text{ if } \tau \text{ is } \textbf{l.p.c.} \text{ at } b, \end{cases}$$

one verifies that the **Dirichlet *m*-function**, where $\alpha = 0$, can be computed via

$$m_{0,\beta_0}(z) = \widetilde{\psi}'_{0,\beta_0}(z,a)/\widetilde{\psi}_{0,\beta_0}(z,a), \quad z \in \mathbb{C}\backslash\mathbb{R}.$$

For other (i.e., **non-Dirichlet**) **b.c.'s**, where $\alpha_0 \neq 0$, use the usual linear fractional transformations (keep $\beta_0$, the **b.c.** at $x = b$, fixed).

Here $\widetilde{\psi}$, $\widetilde{\psi}'$ denote precisely the generalized boundary values we introduced before.

# The Bessel Operator on $(0, \infty)$

## Example (Bessel Operator).

Let $a = 0,\ b = \infty$,

$$p(x) = r(x) = 1,\ q(x) := q_\gamma(x) = \frac{\gamma^2 - (1/4)}{x^2}, \quad \gamma \in [0,1),\ x \in (0,\infty).$$

Then $\tau_\gamma = -d^2/dx^2 + \big[\gamma^2 - (1/4)\big]x^{-2}$, $\gamma \in [0,1)$, $x \in (0,\infty)$, is in the limit circle case at the endpoint 0 and in the limit point case at $\infty$. It suffices to focus on the generalized boundary values at the singular endpoint $x = 0$. To this end we introduce principal and nonprincipal solutions $u_{0,\gamma}(0,\,\cdot\,)$ and $\widehat{u}_{0,\gamma}(0,\,\cdot\,)$ of $\tau_\gamma u = 0$ by

$$u_{0,\gamma}(0,x) = x^{(1/2)+\gamma},\ \gamma \in [0,1),\ x \in (0,\infty),$$

$$\widehat{u}_{0,\gamma}(0,x) = \begin{cases} (2\gamma)^{-1} x^{(1/2)-\gamma}, & \gamma \in (0,1), \\ x^{1/2}\ln(1/x), & \gamma = 0; \end{cases} \quad x \in (0,\infty).$$

# The Bessel Operator on $(0,\infty)$ (contd.)

## Example (Bessel Operator (contd.)).

The generalized boundary values for $g \in \operatorname{dom}(T_{max,\gamma})$ (the maximal operator associated with $\tau_\gamma$) are then of the form

$$
\begin{aligned}
\widetilde{g}(0) &= -W(u_{0,\gamma}(0,\,\cdot\,), g)(0) \\
&= \begin{cases} \lim_{x\downarrow 0} g(x)/\big[(2\gamma)^{-1}x^{(1/2)-\gamma}\big], & \gamma \in (0,1), \\ \lim_{x\downarrow 0} g(x)/\big[x^{1/2}\ln(1/x)\big], & \gamma = 0, \end{cases} \\
\widetilde{g}\,'(0) &= W(\widehat{u}_{0,\gamma}(0,\,\cdot\,), g)(0) \\
&= \begin{cases} \lim_{x\downarrow 0} \big[g(x) - \widetilde{g}(0)(2\gamma)^{-1}x^{(1/2)-\gamma}\big]/x^{(1/2)+\gamma}, & \gamma \in (0,1), \\ \lim_{x\downarrow 0} \big[g(x) - \widetilde{g}(0)x^{1/2}\ln(1/x)\big]/x^{1/2}, & \gamma = 0. \end{cases}
\end{aligned}
$$

# The Bessel Operator on $(0,\infty)$ (contd.)

**Theorem (Bessel operator *m*-function)**

For the (Dirichlet-type) *m*-function one obtains the **Nevanlinna–Herglotz** fct.

$$m_0(z;\gamma) = \begin{cases} -e^{-i\pi\gamma} 2^{-2\gamma-1} \gamma^{-1} [\Gamma(1-\gamma)/\Gamma(1+\gamma)] z^{\gamma}, & \gamma \in (0,1), \\ i(\pi/2) + \ln(2) - \gamma_E - 2^{-1}\ln(z), & \gamma = 0, \end{cases}$$
$$z \in \mathbb{C}\backslash[0,\infty).$$

Here $\gamma_E = 0.57721\dots$ represents Euler's constant, and $\Gamma(\,\cdot\,)$ is the Gamma fct.

**Theorem (Bessel operator *m*-function, contd.)**

In the limit point case where $\gamma \geq 1$, one obtains

$$m_0(z;\gamma) = \begin{cases} -C_\gamma e^{-i\pi\gamma}(2/\pi)\sin(\pi\gamma) z^{\gamma}, & \gamma \in [1,\infty)\backslash\mathbb{N}, \\ C_0(2/\pi) z^n [i - (1/\pi)\ln(z)], & \gamma \in \mathbb{N}, \end{cases} \quad z \in \mathbb{C}\backslash[0,\infty).$$

Thus, the limit point case, $\gamma \geq 1$ naturally leads to a **generalized Nevanlinna–Herglotz** function $m_0(\,\cdot\,;\gamma)$.

# The Legendre Operator on $(-1,1)$

## Example (Legendre Operator).

Let $a=-1$, $b=1$,

$$p(x)=1-x^2,\ r(x)=1,\ q(x)=0, \quad x\in(-1,1).$$

Then $\tau_L=-(d/dx)(1-x^2)(d/dx)$, $x\in(-1,1)$, is in the limit circle case and singular at both endpoints $\pm1$. Principal and nonprincipal solutions $u_{\pm1,L}(0,\,\cdot\,)$ and $\widehat{u}_{\pm1,L}(0,\,\cdot\,)$ of $\tau_L u=0$ at $\pm1$ are then given by

$$u_{\pm1,L}(0,x)=1, \quad \widehat{u}_{\pm1,L}(0,x)=2^{-1}\ln((1-x)/(1+x)), \quad x\in(-1,1).$$

The **generalized boundary values** for $g\in\operatorname{dom}(T_{max,L})$ (the maximal operator associated with $\tau_L$) are then of the form

$$\begin{aligned}\widetilde{g}(\pm1)&=-W(u_{\pm1,L}(0,\,\cdot\,),g)(\pm1)\\&=-(pg')(\pm1)=\lim_{x\to\pm1}g(x)/\big[2^{-1}\ln((1-x)/(1+x))\big],\\\widetilde{g}\,'(\pm1)&=W(\widehat{u}_{\pm1,L}(0,\,\cdot\,),g)(\pm1)\\&=\lim_{x\to\pm1}\big[g(x)-\widetilde{g}(\pm1)2^{-1}\ln((1-x)/(1+x))\big].\end{aligned}$$

# The Legendre Operator on $(-1,1)$ (contd.)

One observes the curious fact that the **Friedrichs** extension $T_{F,L}$ of $T_{min,L}$ (the minimal operator associated with $\tau_L$) then satisfies the boundary conditions

$$\boxed{(pg')(-1) = (pg')(1) = 0,}$$

which resembles the **Neumann** (and **not** the **Dirichlet**) boundary conditions in the context of a regular Sturm–Liouville differential expression on the interval $[-1,1]$. However, since $\tau_L$ is singular at both endpoints $\pm 1$, this represents no conundrum.

**While this is well-known to experts, I will not lie, this fact served as one of the prime motivations to write our paper on this topic!**

In addition, we note that the spectrum of $T_{F,L}$ may be computed explicitly,

$$\sigma(T_{F,L}) = \{n^2 - n\}_{n\in\mathbb{N}}.$$

# The Legendre Operator on $(-1,1)$ (contd.)

## Theorem (Legendre operator *m*-function)

For the (Dirichlet-type) *m*-function one obtains the **Nevanlinna–Herglotz** fct.

$$m_{0,L}(z) = -\frac{1}{2}\big[\pi \cot(\nu(z)\pi) + \gamma_E + 2\psi(1+\nu(z))\big], \quad z \in \rho(T_{F,L}),$$

where we abbreviated

$$\nu(z) := 2^{-1}\big[-1 + (1+4z)^{1/2}\big],$$

and where

$$\psi(z) = \Gamma'(z)/\Gamma(z), \quad z \in \mathbb{C}\backslash\mathbb{N}_0,$$

denotes the Digamma function.

**Now prove from scratch that this is indeed a Nevanlinna–Herglotz fct.!!!!!**

**Note.** $\nu(z) := 2^{-1}\big[-1 + (1+4z)^{1/2}\big]$ is indeed a **Nevanlinna–Herglotz** function.

# The Legendre Operator on $(-1,1)$ (contd.)

Proving this Nevanlinna–Herglotz property of $m_{0,L}(z)$ is tricky: Consider

$$-\pi \cot(z\pi) = \sum_{n\in\mathbb{Z}} \left[\frac{1}{n-z} - \frac{n\pi^2}{n^2\pi^2+1}\right], \quad z \in \mathbb{C}\backslash\mathbb{Z},$$

and

$$\psi(1+z) = -\gamma_E + \sum_{n\in\mathbb{N}} \left[\frac{1}{n} - \frac{1}{n+z}\right], \quad z \in \mathbb{C}\backslash(-\mathbb{N})$$

Then

$$\begin{aligned} m_{0,L}(z) &= -(\pi/2)\cot(\nu(z)\pi) - \gamma_E - \psi(1+\nu(z)) \\ &= \frac{1}{2}\sum_{n\in\mathbb{Z}} \left[\frac{1}{n-\nu(z)} - \frac{n\pi^2}{n^2\pi^2+1}\right] + \sum_{n\in\mathbb{N}} \left[\frac{1}{n+\nu(z)} - \frac{1}{n}\right] \end{aligned}$$

# The Legendre Operator on $(-1,1)$ (contd.)

$$
\begin{aligned}
&= -\frac{1}{2\nu(z)} + \frac{1}{2}\sum_{n\in\mathbb{N}}\left[\frac{1}{n-\nu(z)} - \frac{n\pi^2}{n^2\pi^2+1}\right] \\
&\quad + \frac{1}{2}\sum_{n\in\mathbb{N}}\left[\frac{-1}{n+\nu(z)} + \frac{n\pi^2}{n^2\pi^2+1}\right] \\
&\quad + \frac{1}{2}\sum_{n\in\mathbb{N}}\left[\frac{1}{n+\nu(z)} - \frac{1}{n}\right] + \frac{1}{2}\sum_{n\in\mathbb{N}}\left[\frac{1}{n+\nu(z)} - \frac{1}{n}\right] \\
&= -\frac{1}{2\nu(z)} + \frac{1}{2}\sum_{n\in\mathbb{N}}\left[\frac{1}{n-\nu(z)} - \frac{1}{n}\right] + \frac{1}{2}\sum_{n\in\mathbb{N}}\left[\frac{1}{n+\nu(z)} - \frac{1}{n}\right] \\
&= -\frac{1}{2\nu(z)} + \frac{1}{2}\sum_{n\in\mathbb{N}}\left[\frac{1}{n-\nu(z)} - \frac{1}{n}\right] + \frac{1}{2}\left[\frac{1}{1+\nu(z)} - 1\right] \\
&\quad + \frac{1}{2}\sum_{n=2}^{\infty}\left[\frac{1}{n+\nu(z)} - \frac{1}{n}\right]
\end{aligned}
$$

# The Legendre Operator on $(-1,1)$ (contd.)

$$= -\frac{1}{2}\frac{1}{\nu(z)[\nu(z)+1]} - \frac{1}{2} + \frac{1}{2}\sum_{n\in\mathbb{N}}\left[\frac{1}{n-\nu(z)} - \frac{1}{n}\right]$$

$$+ \frac{1}{2}\sum_{n\in\mathbb{N}}\left[\frac{1}{n+1+\nu(z)} - \frac{1}{n}\right] + \frac{1}{2}\sum_{n\in\mathbb{N}}\left[\frac{1}{n} - \frac{1}{n+1}\right]$$

$$= -\frac{1}{2z} - \frac{1}{2} + \frac{1}{2}\sum_{n\in\mathbb{N}}\frac{1}{n(n+1)}$$

$$+ \frac{1}{2}\sum_{n\in\mathbb{N}}\left[\frac{1}{n+2^{-1}-2^{-1}(1+4z)^{1/2}} - \frac{1}{n}\right]$$

$$+ \frac{1}{2}\sum_{n\in\mathbb{N}}\left[\frac{1}{n+2^{-1}+2^{-1}(1+4z)^{1/2}} - \frac{1}{n}\right]$$

$$= -\frac{1}{2z} + \sum_{n\in\mathbb{N}}\left[\frac{n+2^{-1}}{(n+2^{-1})^2-4^{-1}-z} - \frac{1}{n}\right]$$

$$\boxed{= -\frac{1}{2z} + \sum_{n\in\mathbb{Z}}\left[\frac{n+2^{-1}}{n(n+1)-z} - \frac{1}{n}\right]}, \quad z\in\mathbb{Z}\backslash\{n(n+1)\}_{n\in\mathbb{N}_0}.$$

# The Legendre Operator on $(-1,1)$ (contd.)

Here we used

$$\sum_{n\in\mathbb{N}} \frac{1}{n(n+1)} = 1,$$

and

$$\nu(z)[\nu(z)+1] = z, \quad z \in \mathbb{C}.$$

Once again, one confirms explicitly that the set of poles of $m_{0,L}(\,\cdot\,)$ coincides with the spectrum of $T_{F,L}$,

$$\sigma(T_{F,L}) = \{n^2 - n\}_{n\in\mathbb{N}}.$$

# The Laguerre (resp., Kummer, or Confluent Hypergeometric) Operator on $(0,\infty)$

## Example (Laguerre Operator).

Let $a = 0$, $b = \infty$,

$$p(x) = p_\beta(x) = x^\beta e^{-x},\ q(x) = 0,\ r(x) := r_\beta(x) = x^{\beta-1} e^{-x}, \quad \beta \in (0,2),\ x \in (0,\infty).$$

Then $\tau_\beta = -x^{1-\beta} e^x \frac{d}{dx} x^\beta e^{-x} \frac{d}{dx}$, $x \in (0,\infty)$, and the underlying Hilbert space is $L^2((0,\infty); x^{\beta-1} e^{-x}\, dx)$. At $x = 0$, $\tau_\beta$ is regular for $\beta \in (0,1)$ and singular for $\beta \in [1,2)$.

For $z \in \mathbb{C}$, solutions to the Kummer equation $\tau_\beta y = zy$ are given by

$$y_{1,\beta}(z,x) = F(-z,\beta;x), \quad \beta \in (0,2),\ z \in \mathbb{C},\ x \in (0,\infty),$$

$$y_{2,\beta}(z,x) = \begin{cases} x^{1-\beta} F(1-\beta-z, 2-\beta; x), & \beta \in (0,2)\backslash\{1\},\ z \in \mathbb{C}, \\ \Gamma(-z) U(-z,1;x), & \beta = 1,\ z \in \mathbb{C}\backslash\{0\}, \\ -\displaystyle\int_1^x dt\, t^{-1} e^t, & \beta = 1,\ z = 0;\ x \in (0,\infty), \end{cases}$$

# The Laguerre Operator on $(0, \infty)$ (contd.)

**Example (Laguerre Operator (contd.)).**

where $F(\,\cdot\,, \cdot\,; \cdot\,)$ (also frequently denoted by ${}_1F_1(\,\cdot\,, \cdot\,; \cdot\,)$ or $M(\,\cdot\,, \cdot\,; \cdot\,)$) denotes the confluent hypergeometric function and $U(\,\cdot\,, 1; \cdot\,)$ represents an associated logarithmic case.

A principal solution of $\tau_\beta u = \lambda u$, $\lambda \leq 0$, at $x = 0$ is given by

$$u_{0,\beta}(\lambda, \,\cdot\,) = \begin{cases} (1-\beta)^{-1} y_{2,\beta}(\lambda, \,\cdot\,), & \beta \in (0,1), \\ -(1-\beta)^{-1} y_{1,\beta}(\lambda, \,\cdot\,), & \beta \in (1,2), \\ y_{1,1}(\lambda, \,\cdot\,), & \beta = 1, \end{cases} \quad \lambda \leq 0, \tag{7.1}$$

and a nonprincipal solution of $\tau_\beta u = \lambda u$ at $x = 0$ is given by

$$\widehat{u}_{0,\beta}(\lambda, \,\cdot\,) = \begin{cases} y_{1,\beta}(\lambda, \,\cdot\,), & \beta \in (0,1), \\ y_{2,\beta}(\lambda, \,\cdot\,), & \beta \in [1,2). \end{cases} \quad \lambda \leq 0. \tag{7.2}$$

# The Laguerre Operator on $(0,\infty)$ (contd.)

## Example (Laguerre Operator (contd.)).

The generalized boundary values for $g \in \operatorname{dom}(T_{max,\beta})$ (the maximal operator associated with $\tau_\beta$) are then of the form

$$\widetilde{g}(0) = -W(u_{0,\beta}(0,\,\cdot\,), g)(0) = \lim_{x\downarrow 0} \frac{g(x)}{\widehat{u}_{0,\beta}(0,x)} = \begin{cases} g(0), & \beta \in (0,1), \\ \lim_{x\downarrow 0} \dfrac{g(x)}{x^{1-\beta}}, & \beta \in (1,2), \\ \lim_{x\downarrow 0} \dfrac{g(x)}{[-\ln(x)]}, & \beta = 1, \end{cases}$$

$$\widetilde{g}\,'(0) = W(\widehat{u}_{0,\beta}(0,\,\cdot\,), g)(0) = \lim_{x\downarrow 0} \frac{g(x) - \widetilde{g}(0)\widehat{u}_{0,\beta}(0,x)}{u_{0,\beta}(0,x)}$$

$$= \begin{cases} \lim_{x\downarrow 0} \dfrac{g(x)-g(0)}{(1-\beta)^{-1}x^{1-\beta}} = \dfrac{0}{0} = \lim_{x\downarrow 0} \dfrac{g'(x)}{x^{-\beta}} = g^{[1]}(0), & \beta \in (0,1), \\ (\beta-1)\lim_{x\downarrow 0}\big[g(x) - \widetilde{g}(0)x^{1-\beta}\big], & \beta \in (1,2), \\ \lim_{x\downarrow 0}\big\{g(x) - \widetilde{g}(0)[-\ln(x)]\big\}, & \beta = 1. \end{cases}$$

# The Laguerre Operator on $(0,\infty)$ (contd.)

**Theorem (Laguerre operator *m*-function)**

For the (Dirichlet-type) *m*-function one obtains the **Nevanlinna–Herglotz** fct.

$$m_{0,\beta}(z) = \begin{cases} \frac{(1-\beta)\Gamma(2-\beta)\Gamma(-z)}{\Gamma(\beta)\Gamma(1-\beta-z)}, & \beta \in (1,2),\ z \in \rho(T_{F,\beta}), \\ -\psi(-z), & \beta = 1,\ z \in \rho(T_{F,1}). \end{cases}$$

Once again, here

$$\psi(z) = \Gamma'(z)/\Gamma(z), \quad z \in \mathbb{C}\backslash\mathbb{N}_0,$$

denotes the Digamma function.

We recall

$$\sigma(T_{F,\beta}) = \begin{cases} \{n+1-\beta\}_{n\in\mathbb{N}_0}, & \beta \in (0,1), \\ \mathbb{N}_0, & \beta \in [1,2). \end{cases}$$