Effective parameters of periodic electromagnetic structures from spatio-temporal Kramers-Kronig relations

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Herglotz-Nevanlinna Theory Applied to Passive, Causal and Active Systems
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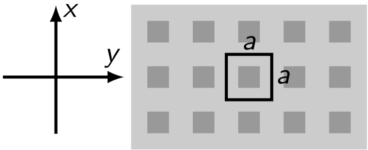
Objective

Propagation of EM waves

in periodic structures : $\varepsilon(x, y)$

(no boundaries)



modeling for all frequency and wavevector

Effective homogeneous parameter

for the propagation of EM waves

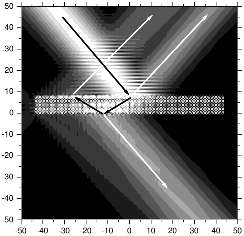
$\rightarrow \varepsilon_{\mathbf{eff}}(\omega, \boldsymbol{k})$

$\rightarrow n_{\mathbf{eff}}(\omega, k) = \sqrt{\varepsilon_{\mathbf{eff}}(\omega, \boldsymbol{k})}$

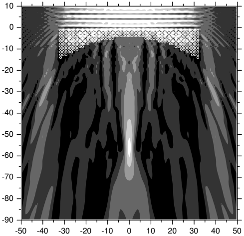
$\varepsilon_{\mathrm{eff}}(\omega, \boldsymbol{k})$

Motivation : composite materials (metamaterials)

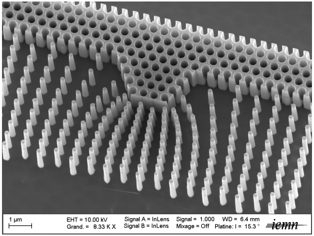
Modeling of unusal effective properties : $n_{\text{eff}} < 1$, $n_{\text{eff}} < 0$, μ_{eff} ...



$n_{\text{eff}} < 0$



$n_{\text{eff}} < 1$



invisibility carpet

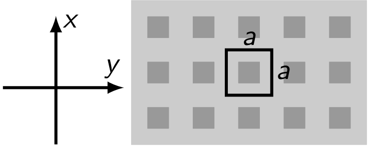
J. Opt. Soc. Am. A **17**, 001012 (2000)
Phys. Rev. B **88**, 115110 (2013)

Objective : modeling of propagation of EM waves

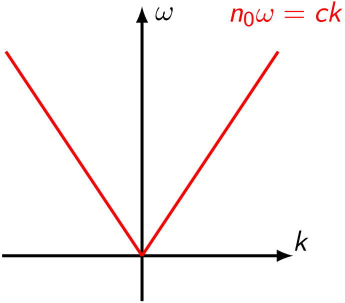
Propagation of EM waves

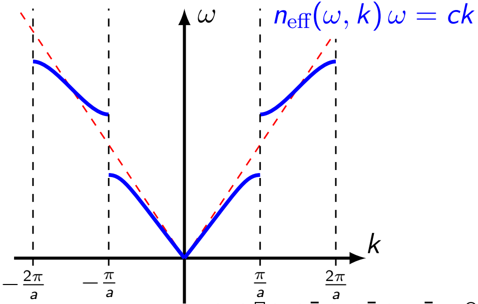
in periodic structures : $\varepsilon(x, y)$

(no boundaries)



Propagation of EM waves is governed by the **dispersion law** : $\omega(k)$



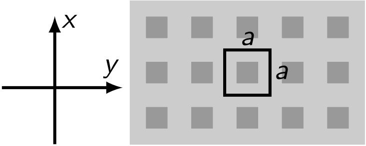


Objective : modeling of the dispersion law

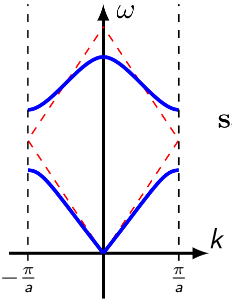
The dispersion law : $\omega(k)$

The effective parameter :

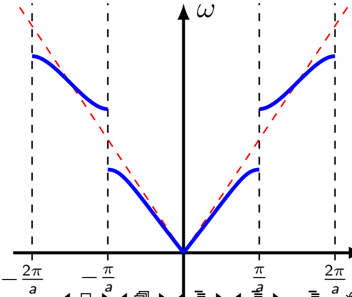
$$n_{\text{eff}}(\omega, k)\,\omega = ck$$



The dispersion law : folded or developed ?



same phase shift
same group velocity
same propagation

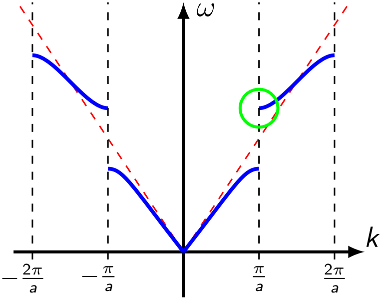


Analytic structure of the developed dispersion law

The complex frequency : $\omega \to \omega + i\eta = \omega$

The complex wavevector : $k \to k + i\xi = k$

Assumption : analyticity of the developed dispersion law



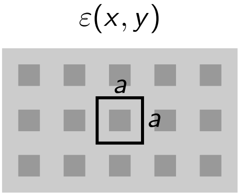
$$n_{\mathrm{eff}}(\omega, k)\,\omega = ck$$

All the information
$n_{\mathrm{eff}}(\omega, k)$ **for** (ω, k) **in** ◯

$\Longleftrightarrow$

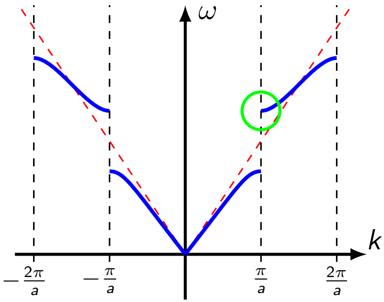
All the information
$n_{\mathrm{eff}}(\omega, k)$ **for all** (ω, k)

Main ideas for the modeling of the dispersion law



dispersion law :

$n_{\text{eff}}(\omega, k)\,\omega = ck$



→ **consider the developed dispersion law**

→ **consider complex frequency and wavevector** (ω, k)

→ **assume effective parameters** $n_{\text{eff}}(\omega, k)$ **analytic of** (ω, k)

→ **use perturbation technique to obtain information in** ○

→ **use analytic continuation (Kramers-Kronig relations) to obtain** $n_{\text{eff}}(\omega, k)$

A motivation

An opportunity to investigate

spatial dispersion (ω, k)

Outline of the presentation

Outline of the presentation

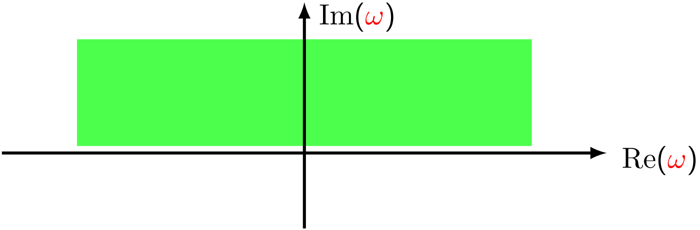
$$\boldsymbol{P}(\boldsymbol{x}, t) = \int_{-\infty}^{t} ds\, \chi(\boldsymbol{x}, t-s)\, \boldsymbol{E}(\boldsymbol{x}, s)$$

Analytic property from time causality

$\chi(\boldsymbol{x}, t) = 0$ **in the domain** $t < 0$

$\Longleftrightarrow$†

$\varepsilon(\boldsymbol{x}, \omega)$, $\boldsymbol{E}(\boldsymbol{x}, \omega)$, $\mathrm{R}(\omega)$... **analytic in the domain** $\mathbf{Im}(\omega) > 0$



† Related to the Paley-Wiener theorem.

Analytic property from causality principle

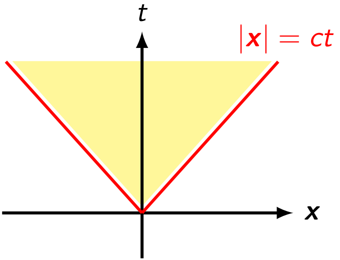
$$\boldsymbol{P}(\boldsymbol{x}, t) = \int_{-\infty}^{t} ds \int_{|\boldsymbol{x}-\boldsymbol{y}|\le ct} \chi(\boldsymbol{x}-\boldsymbol{y}, t-s)\, \boldsymbol{E}(\boldsymbol{y}, s)$$

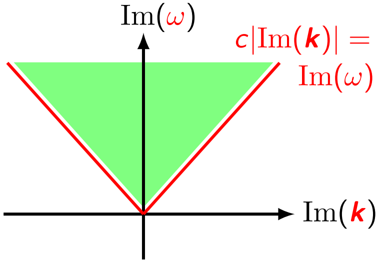
Analytic property from space-time causality

light cone : $\chi(\boldsymbol{x}, t) = 0,\ G(\boldsymbol{x}, t) = 0$ **in the domain** $t < |\boldsymbol{x}|/c$

$$\Longleftrightarrow^{\dagger}$$

$\varepsilon(\boldsymbol{k}, \omega),\ \boldsymbol{E}(\boldsymbol{x}, \boldsymbol{k}, \omega)$**... analytic in the cone** $\mathrm{Im}(\omega) - c|\mathrm{Im}(\boldsymbol{k})| > 0$





† Related to the Paley-Wiener theorem.

Time harmonic Maxwell's equations :

$$\nabla \times \boldsymbol{H}(\boldsymbol{x},\omega) = -i\omega\varepsilon(\boldsymbol{x},\omega)\boldsymbol{E}(\boldsymbol{x},\omega)\,,$$

$$\nabla \times \boldsymbol{E}(\boldsymbol{x},\omega) = i\omega\mu_0\boldsymbol{H}(\boldsymbol{x},\omega)\,.$$

Periodicity and Bloch decomposition : $\nabla \longrightarrow \nabla + i\boldsymbol{k}$

$$[\nabla + i\boldsymbol{k}] \times \boldsymbol{H}(\boldsymbol{x},\boldsymbol{k},\omega) = -i\omega\varepsilon(\boldsymbol{x},\omega)\boldsymbol{E}(\boldsymbol{x},\boldsymbol{k},\omega)\,,$$

$$[\nabla + i\boldsymbol{k}] \times \boldsymbol{E}(\boldsymbol{x},\boldsymbol{k},\omega) = i\omega\mu_0\boldsymbol{H}(\boldsymbol{x},\boldsymbol{k},\omega)\,.$$

The fields $\boldsymbol{E}(\boldsymbol{x},t)$ **can be expressed from the dispersion law** $\omega(\boldsymbol{k})$ **or** $\boldsymbol{k}(\omega)$ **:** $\boldsymbol{E}(\boldsymbol{x},t) = \int d\omega d\boldsymbol{k}\,\exp[i\boldsymbol{k}\cdot\boldsymbol{x} - i\omega t]\hat{\boldsymbol{E}}(\boldsymbol{k},\omega(\boldsymbol{k}))$

space-time causality : analytic if $\mathrm{Im}(\omega) - c|\mathrm{Im}(\boldsymbol{k})| > 0$

$\rightarrow$ $\omega(\boldsymbol{k})$ **or** $\boldsymbol{k}(\omega)$ **have analytic properties** [1, 2]

1. H. Knörrer and E. Trubovitz, Comment. Math. Helvetici **65**, 114-149 (1990).
2. http ://arxiv.org/abs/1807.01658 (Editors V. Markel and I. Tsukerman)

Analytic property of $n_{\text{eff}}(\omega, k)$

- **The dispersion law $\omega(\boldsymbol{k})$ or $\boldsymbol{k}(\omega)$ has the analytic property related to the space-time causality**
- **The effective parameters $n_{\text{eff}}(\omega, k)$ are derived from the dispersion law**

Assumption* : $n_{\text{eff}}(\omega, k)$ analytic if $\mathbf{Im}(\omega) - c|\mathbf{Im}(k)| > 0$

Consequence (related to the Paley-Wiener theorem) :

$$n_{\text{eff}}(\omega, k) = \int_0^{\infty} dt \int_{|\boldsymbol{x}|\leq ct} d\boldsymbol{x} \exp[i\omega t - k \cdot \boldsymbol{x}]\chi_{\text{eff}}(\boldsymbol{x}, t)$$

$\longrightarrow$ **True in 1D***†

† Phys. Rev. B **88**, 165104 (2013).

Outline of the presentation

Kramers-Kronig relations

$$\varepsilon(\omega) = \varepsilon_0 + \int_0^\infty dt \exp[i\omega t]\chi(t)\,, \qquad \sigma(\nu) = \frac{\mathrm{Im}[\,\nu\,\varepsilon(\nu)\,]}{\pi} > 0\,.$$

use of causality / analyticity : $\varepsilon(\omega) - \varepsilon_0 = \widehat{\chi} = \widehat{\theta} * \widehat{\chi} = \dfrac{1}{\omega} * \widehat{\chi}$ ...

"Kramers-Kronig relations" for $\mathrm{Im}(\omega) > 0$
$\longrightarrow$ **"representation of Herglotz-Nevanlinna functions"**

$$\varepsilon(\omega) = \varepsilon_0 - \int_{\mathbb{R}} d\nu\, \frac{\sigma(\nu)}{\omega^2 - \nu^2}\,.$$

Superposition of elementary resonances[†]**:** $\varepsilon(\omega) = \varepsilon_0 - \dfrac{\Omega^2}{\omega^2 - \nu^2}$

Simple models for elementary resonances :

$\rightarrow$ **elastically bound electron**[†]

$\rightarrow$ **coupling of EM waves with quantized atom**

$\rightarrow$ **any causal and passive system...**

Kramers-Kronig relations with spatial dispersion

$$\varepsilon(\omega, \boldsymbol{k}) = \varepsilon_0 + \int_0^\infty dt \int_{|\boldsymbol{x}|\leq ct} d\boldsymbol{x}\, \exp[i\omega t - \boldsymbol{k}\cdot\boldsymbol{x}]\chi(\boldsymbol{x}, t)\,,$$

$\rightarrow$ **introduction of** $\sigma(\nu, \boldsymbol{\kappa}) = \dfrac{\mathrm{Im}[\,\nu\,\varepsilon(\nu, \boldsymbol{\kappa})\,]}{\pi}$ **and use of causality**

Different results depending on $\boldsymbol{x}\,,\ \boldsymbol{k} \in \mathbb{R}\,,\ \mathbb{R}^2\,,\ \mathbb{R}^3$ **:**

1D : $\omega[\varepsilon(\omega, k) - \varepsilon_0] = -\dfrac{i}{\pi c}\displaystyle\int_{\mathbb{R}} d\nu \int_{\mathbb{R}} d\kappa\, \frac{\sigma(\nu, \kappa)}{(\omega-\nu)^2/c^2 - (k-\kappa)^2}\,.$

2D : $\omega[\varepsilon(\omega, k) - \varepsilon_0] = \dfrac{1}{2\pi c}\displaystyle\int_{\mathbb{R}} d\nu \int_{\mathbb{R}} d\kappa\, \frac{\sigma(\nu, \kappa)}{[(\omega-\nu)^2/c^2 - (k-\kappa)^2]^{3/2}}\,.$

3D : $\omega[\varepsilon(\omega, k) - \varepsilon_0] = \dfrac{i}{\pi^2 c}\displaystyle\int_{\mathbb{R}} d\nu \int_{\mathbb{R}} d\kappa\, \frac{\sigma(\nu, \kappa)}{[(\omega-\nu)^2/c^2 - (k-\kappa)^2]^{2}}\,.$

Kramers-Kronig relations with spatial dispersion

"Kramers-Kronig relations" for $n_{\text{eff}}(\omega, k)$, x and $k \in \mathbb{R}$

$$\omega[n_{\text{eff}}(\omega, k) - \varepsilon_0] = -\frac{i}{\pi c} \int_{\mathbb{R}} d\nu \int_{\mathbb{R}} d\kappa \, \frac{\sigma(\nu, \kappa)}{(\omega - \nu)^2/c^2 - (k - \kappa)^2} \,,$$

where :

$$\sigma(\nu, \kappa) = \frac{\mathrm{Im}[\, \nu \, n_{\text{eff}}(\nu, \kappa)\,]}{\pi} > 0 \,.$$

Superposition of elementary convolutions with $\dfrac{1}{\omega^2/c^2 - k^2}$

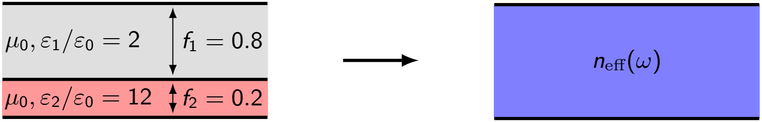
Simple model for elementary resonances :

$\rightarrow$ **convolution with the free scalar EM Green's function**

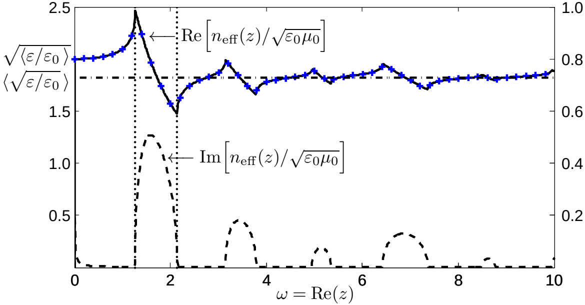
$\rightarrow$ **may be not a coincidence...**

$\rightarrow$ **related to a "Herglotz-Nevanlinna representation" ?**

Numerical check of the Kramers-Kronig relation in 1D



The effective index of a multilayer



$+$ **Kramers-Kronig relations ;** $-$ **exact retrieval expression**

Phys. Rev. B **88**, 165104 (2013)

Outline of the presentation

Perturbation technique

Kramers-Kronig relations for ω and k :

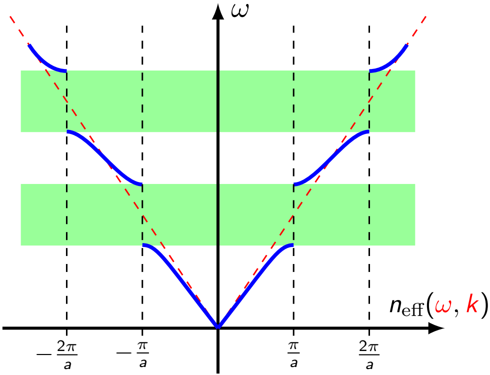
$$\omega[n_{\text{eff}}(\omega, k) - n_0(\omega)] = -\frac{i}{\pi c} \int_{\mathbb{R}} d\nu \int_{\mathbb{R}} d\kappa \, \frac{\sigma(\nu, \kappa)}{(\omega - \nu)^2/c^2 - (k - \kappa)^2} \, ,$$

where

$$\sigma(\nu, \kappa) = \frac{\mathrm{Im}[\, \nu \, n_{\text{eff}}(\nu, \kappa)\,]}{\pi} \neq 0$$

is **attenuation**

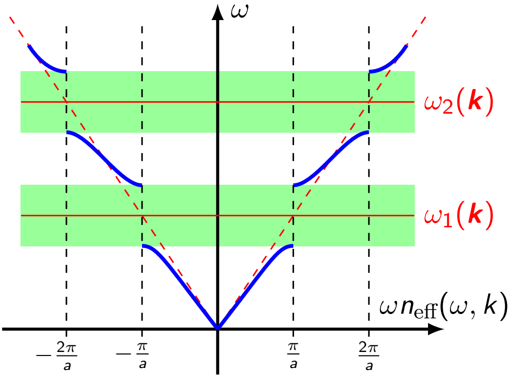
in **stop bands**



Perturbation technique

Small contrast (perturbation) : stop band width $\ll \omega_p(k)$

$$\sigma(\omega, k) = \frac{\mathrm{Im}[\,\omega\, n_{\mathrm{eff}}(\omega, k)\,]}{\pi} \approx \sum_p \delta\left[\omega^2 - \omega_p^2(k)\right] \Omega_p^2(k)\,.$$



Perturbation technique :

$\rightarrow \omega_p^2(k)$

$\rightarrow \Omega_p^2(k)$

Perturbation technique

Kramers-Kronig relations for ω and k :

$$\omega[n_{\text{eff}}(\omega, k) - n_0(\omega)] = -\frac{i}{\pi c}\int_{\mathbb{R}} d\nu \int_{\mathbb{R}} d\kappa \, \frac{\sigma(\nu, \kappa)}{(\omega - \nu)^2/c^2 - (k - \kappa)^2} \, .$$

Small contrast and perturbation technique :

$$\sigma(\nu, \kappa) \approx \sum_p \delta\left[\nu^2 - \omega_p^2(\kappa)\right] \Omega_p^2(\kappa) \, .$$

Resulting expression :

$$n_{\text{eff}}(\omega, k) - n_0(\omega) \approx -\sum_p \frac{\Omega_p^2(k)}{\omega^2 - \omega_p^2(k)} \, .$$

Outline of the presentation

Application to the 1D case

Approached expression :

$$n_{\text{eff}}(\omega, k) - n_0(\omega) \approx -\sum_p \frac{\Omega_p^2(k)}{\omega^2 - \omega_p^2(k)} \,.$$

with :

$$n_0(\omega) = \left[\int_0^a dx \, \frac{\varepsilon_0}{\varepsilon(x,\omega)}\right]^{-1/2}$$

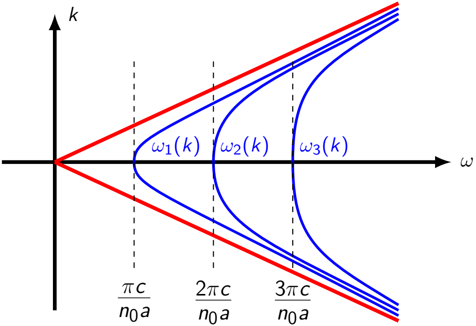
$$\omega_p^2(k) = \frac{c^2}{n_0^2(\omega_p(k))} \left[p^2\pi^2/a^2 + k^2\right]$$

$$\Omega_p^2(k) = \frac{c^2}{n_0^2(\omega_p(k))} \frac{\left[p^2\pi^2/a^2 + k^2\right]^2}{p^2\pi^2/a^2} \alpha_p(\omega_p(k))\, \alpha_{-p}(\omega_p(k))$$

$$\alpha_p(\omega) = n_0^2(\omega) \left[\int_0^a dx \, \frac{\varepsilon_0}{\varepsilon(x,\omega)} \exp[i2p\pi x/a]\right]$$

Application to the 1D case

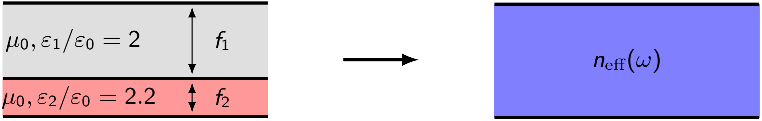
Approached expression : $n_{\text{eff}}(\omega, k) - n_0(\omega) \approx -\sum_p \frac{\Omega_p^2(k)}{\omega^2 - \omega_p^2(k)}$

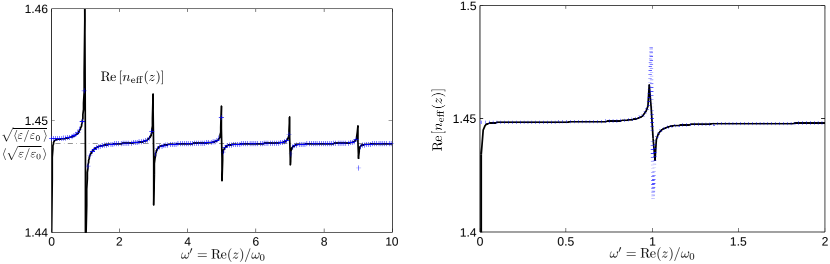


Case without dispersion : $\omega_p^2(k) = [p^2\pi^2/a^2 + k^2]c^2/n_0^2 \longrightarrow$ $n_{\text{eff}}(\omega, k)$ **as a sum of hydrodynamical model resonances :**

$$\varepsilon = \varepsilon_0 - \frac{\Omega^2}{\omega^2 - \omega_0^2 - v^2 k^2}$$

Application to the 1D case





+ model ; − exact retrieval expression

Y. Liu, PhD thesis, Aix-Marseille university (2013)

Outline of the presentation

Imaginary part of the permeability

Kramers-Kronig relations for $\mathrm{Im}(\omega) > 0$
$\longrightarrow$ representation of Herglotz-Nevanlinna functions

$$\mu(\omega) = \mu_0 - \frac{1}{\pi} \int_{\mathbb{R}} d\nu \, \frac{\mathrm{Im}[\nu\mu(\nu)]}{\omega^2 - \nu^2} \, .$$

At the nul frequency (static) :

$$\mu(0) = \mu_0 + \frac{2}{\pi} \int_0^\infty d\nu \, \frac{\mathrm{Im}[\nu\mu(\nu)]}{\nu^2}$$

Paramagnetic media: $\mu(0) - \mu_0 = \dfrac{2}{\pi} \displaystyle\int_0^\infty d\nu \, \frac{\mathrm{Im}[\nu\mu(\nu)]}{\nu^2} \; > \; 0 \, .$

Diamagnetic media: $\mu(0) - \mu_0 = \dfrac{2}{\pi} \displaystyle\int_0^\infty d\nu \, \frac{\mathrm{Im}[\nu\mu(\nu)]}{\nu^2} \; < \; 0 \, .$

Imaginary part of the permeability : positive or negative ?

Anomalies in the dispersion of effective permeability

Questions on the sign of the imaginary part of $\omega\mu_{\mathbf{eff}}(\omega)^{\dagger}$

PHYSICAL REVIEW E **78**, 026608 (2008)

Can the imaginary part of permeability be negative?

PHYSICAL REVIEW B **83**, 081102(R) (2011)

Restoring the physical meaning of metamaterial constitutive parameters

PHYSICAL REVIEW B **83**, 165119 (2011)

Examining the validity of Kramers-Kronig relations for the magnetic permeability

$\rightarrow$ test with the effective parameters of a 1D system

†The Kramers-Kronig relations are modified for $\mu(\omega)$ in the book by Landau and Lifshitz, *Electrodynamics of continuous media*.

Spatial dispersion and permeability

Maxwell's equations in magneto-dielectric media

$$-\boldsymbol{\nabla} \times \frac{1}{\omega\mu(\omega)} \boldsymbol{\nabla} \times \boldsymbol{E}(\boldsymbol{x}) + \omega\,\varepsilon(\omega) \boldsymbol{E}(\boldsymbol{x}) = 0 \qquad \text{(source free)}$$

$\rightarrow$ **can be written**

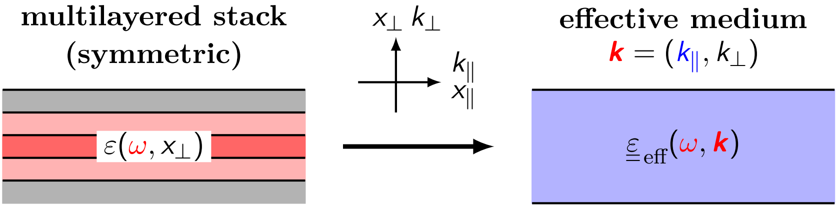
$$-\boldsymbol{\nabla} \times \frac{1}{\omega\mu_0} \boldsymbol{\nabla} \times \boldsymbol{E}(\boldsymbol{x}) - \boldsymbol{\nabla} \times \left[\frac{1}{\omega\mu(\omega)} - \frac{1}{\omega\mu_0}\right] \boldsymbol{\nabla} \times \boldsymbol{E}(\boldsymbol{x}) + \omega\,\varepsilon(\omega) \boldsymbol{E}(\boldsymbol{x}) = 0$$

and, in a homogeneous medium, $\boldsymbol{\nabla}\times \longleftrightarrow i\boldsymbol{k}\times$

$$-\boldsymbol{\nabla} \times \frac{1}{\omega\mu_0} \boldsymbol{\nabla} \times \boldsymbol{E}(\boldsymbol{x}) + \boldsymbol{k} \times \left[\frac{1}{\omega\mu(\omega)} - \frac{1}{\omega\mu_0}\right] \boldsymbol{k} \times \boldsymbol{E}(\boldsymbol{x}) + \omega\,\varepsilon(\omega) \boldsymbol{E}(\boldsymbol{x}) = 0\,.$$

Permittivity $\underline{\underline{\varepsilon}}(\omega, \boldsymbol{k})$ **with spatial dispersion** $(\omega, \boldsymbol{k})$ **defines permeability** $\mu(\omega)$ **:**

$$\omega\,\underline{\underline{\varepsilon}}_{\text{eff}}(\omega, \boldsymbol{k}) = \omega\,\varepsilon_{\text{eff}}(\omega) + \boldsymbol{k} \times \left[\frac{1}{\omega\mu_{\text{eff}}(\omega)} - \frac{1}{\omega\mu_0}\right] \boldsymbol{k}\times$$



The effective permittivity with spatial dispersion $(\omega, \boldsymbol{k})$ is

$$\omega \underline{\underline{\varepsilon}}_{\text{eff}}(\omega, \boldsymbol{k}) = \omega \varepsilon_{\text{eff}}(\omega, k_\parallel) + \boldsymbol{k} \times \left[\frac{1}{\omega \mu_{\text{eff}}(\omega, k_\parallel)} - \frac{1}{\omega \mu_0} \right] \boldsymbol{k} \times$$

where, for $\xi_{\text{eff}}(\omega, k_\parallel) = \varepsilon_{\text{eff}}(\omega, k_\parallel),\ \mu_{\text{eff}}(\omega, k_\parallel)$ **:**

$$\xi_{\text{eff}}(\omega, k_\parallel) = \begin{bmatrix} \xi_\parallel(\omega, k_\parallel) & 0 & 0 \\ 0 & \xi_\parallel(\omega, k_\parallel) & 0 \\ 0 & 0 & \xi_\perp(\omega, k_\parallel) \end{bmatrix}.$$

Effective parameters of a multilayered stack

Four effective parameters :

$$\varepsilon_{\parallel}(\omega, k_{\parallel}), \qquad \varepsilon_{\perp}(\omega, k_{\parallel}), \qquad \mu_{\parallel}(\omega, k_{\parallel}), \qquad \mu_{\perp}(\omega, k_{\parallel})$$

Four parameters in the transfer matrices s and p :

$$\begin{bmatrix} \cos\left[k_{\perp}^{s,p}(\omega, k_{\parallel})\right] & \left[Z^{s,p}(\omega, k_{\parallel})\right]^{-1} \sin\left[k_{\perp}^{s,p}(\omega, k_{\parallel})\right] \\ -Z^{s,p}(\omega, k_{\parallel}) \sin\left[k_{\perp}^{s,p}(\omega, k_{\parallel})\right] & \cos\left[k_{\perp}^{s,p}(\omega, k_{\parallel})\right] \end{bmatrix}$$

Exact retrieval method (no approximation) :

$$\omega\, \varepsilon_{\parallel}(\omega, k_{\parallel}) = k_{\perp}^{p}(\omega, k_{\parallel})/Z^{p}(\omega, k_{\parallel})$$

$$\omega\, \mu_{\parallel}(\omega, k_{\parallel}) = k_{\perp}^{s}(\omega, k_{\parallel}) Z^{s}(\omega, k_{\parallel})$$

$$\frac{1}{\omega \varepsilon_{\perp}(\omega, k_{\parallel})} = \frac{k_{\perp}^{p}(\omega, k_{\parallel}) Z^{p}(\omega, k_{\parallel}) - k_{\perp}^{s}(\omega, k_{\parallel}) Z^{s}(\omega, k_{\parallel})}{k_{\parallel}{}^{2}}$$

$$\frac{1}{\omega \mu_{\perp}(\omega, k_{\parallel})} = \frac{k_{\perp}^{s}(\omega, k_{\parallel})/Z^{s}(\omega, k_{\parallel}) - k_{\perp}^{p}(\omega, k_{\parallel})/Z^{p}(\omega, k_{\parallel})}{k_{\parallel}{}^{2}}$$

Properties of the effective parameters

In the domain $\mathrm{Im}(\omega) - c|\mathrm{Im}(k_{\parallel})| > 0$

The four effective parameters are $(\omega, k_{\parallel})$-analytic† :

$$\omega\varepsilon_{\parallel}(\omega, k_{\parallel})\,, \qquad \omega\mu_{\parallel}(\omega, k_{\parallel})\,, \qquad \frac{1}{\omega\varepsilon_{\perp}(\omega, k_{\parallel})}\,, \qquad \frac{1}{\omega\mu_{\perp}(\omega, k_{\parallel})}\,.$$

The absence of Bloch modes‡ implies :

$$\mathrm{Im}k_{\perp}^{s}(\omega, k_{\parallel}) > 0 \qquad \mathrm{Im}k_{\perp}^{p}(\omega, k_{\parallel}) > 0$$

$\mathrm{Im}\left[\omega\varepsilon(\omega, x_{\perp})\right] - c|\mathrm{Im}(k_{\parallel})| > 0$ of the permittivity implies† :

$$\mathrm{Im}\left[\omega\varepsilon_{\parallel}(\omega, k_{\parallel})\right] > 0\,, \qquad \mathrm{Im}\left[\omega\varepsilon_{\perp}(\omega, k_{\parallel})\right] > 0\,,$$

$$\mathrm{Re}Z^{s}(\omega, k_{\parallel}) > 0\,, \qquad \mathrm{Re}Z^{p}(\omega, k_{\parallel}) > 0\,.$$

$\longrightarrow$ **No condition on** $\mathrm{Im}\left[\omega\mu_{\parallel}(\omega, k_{\parallel})\right]$ **and** $\mathrm{Im}\left[\omega\mu_{\perp}(\omega, k_{\parallel})\right]$.

† Phys. Rev. B **88**, 165104 (2013)

‡ J. Phys. A : Math. Gen. **33**, 006223 (2000)

The imaginary part of the effective permeability

Let $\mathrm{Im}(\omega) = \eta > 0$ **be fixed : from the** $(\omega, k_{\|})$**-analyticity**

$$\int_{\mathbb{R}+i\eta} d\omega \left[\frac{1}{\omega \mu_{\rm eff}(\omega, k_{\|})} - \frac{1}{\omega \mu_0} \right] = 0 \, .$$

Taking the limit $\eta \downarrow 0$

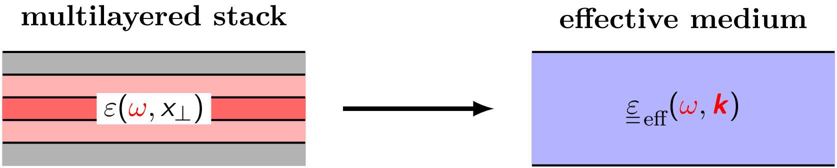
$$\mathcal{PV} \int_{\mathbb{R}} d\omega \left[\frac{1}{\omega \mu_{\rm eff}(\omega, k_{\|})} - \frac{1}{\omega \mu_0} \right] = i\pi \left[\frac{1}{\mu_{\rm eff}(0, k_{\|})} - \frac{1}{\mu_0} \right] \, .$$

Since $\mu_{\rm eff}(0, k_{\|}) = \mu_0$:

$$\mathrm{Im} \int_0^{\infty} d\omega \left[\frac{1}{\omega \mu_{\rm eff}(\omega, k_{\|})} - \frac{1}{\omega \mu_0} \right] = 0 \implies \int_0^{\infty} d\omega \frac{\mathrm{Im}\left[\omega \mu_{\rm eff}(\omega, k_{\|})\right]}{\left|\omega \mu_{\rm eff}(\omega, k_{\|})\right|^2} = 0 \, .$$

$\boxed{\omega \mu_{\rm eff}(\omega, k_{\|}) \textbf{ is not a Herglotz function} \longrightarrow \omega \underline{\underline{\varepsilon}}_{\rm eff}(\omega, k_{\|}) \textbf{ is !}}$

And the (original) system is passive...



The system is passive : $\mathrm{Im}\left[\omega\varepsilon(\omega, x_\perp)\right] \geq \mathrm{Im}(\omega\varepsilon_0) \geq 0$

The imaginary part of $\omega\underline{\underline{\varepsilon}}_{\text{eff}}(\omega, \boldsymbol{k})$ is positive :

$$\mathrm{Im}\left[\omega\varepsilon_{\text{eff}}(\omega, k_\parallel) + \boldsymbol{k} \times \left(\frac{1}{\omega\mu_{\text{eff}}(\omega, k_\parallel)} - \frac{1}{\omega\mu_0}\right)\boldsymbol{k}\times\right] \geq \mathrm{Im}(\omega\varepsilon_0) \geq 0\,,$$

while $\mathrm{Im}\left[\omega\mu_{\text{eff}}(\omega, k_\parallel)\right]$ takes both positive and negative values since

$$\int_0^\infty d\omega \frac{\mathrm{Im}\left[\omega\mu_{\text{eff}}(\omega, k_\parallel)\right]}{\left|\omega\mu_{\text{eff}}(\omega, k_\parallel)\right|^2} = 0\,.$$

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Thank you