Analytic continuation problems via reproducing kernel Hilbert spaces

Narek Hovsepyan

joint work with

Yury Grabovsky

BIRS

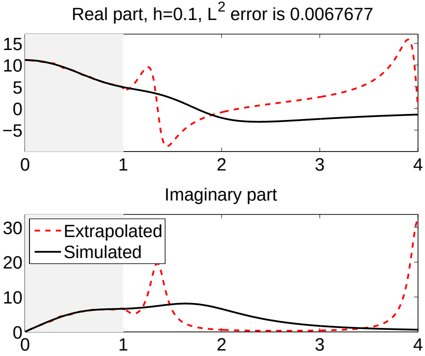
October 8, 2019

Motivation

- Extrapolation of complex electromagnetic permittivity from a given band of frequencies

- Determination of geometric features of microstructure of a composite based on measurements of its effective properties

- Recovery of a signal corrupted by a low-pass convolution filter



Problem formulation

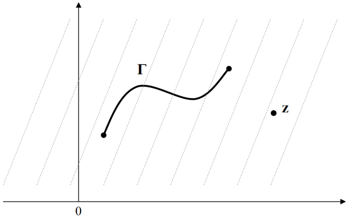
- F, analytic in Ω, is measured on a curve $\Gamma \subset \Omega$ (or $\Gamma \subset \partial\Omega$) with relative error ϵ w.r.t. $\|F\|_\Gamma$

- Goal: quantify the degree to which analytic continuation from Γ to Ω is possible

- F_1, F_2 analytic continuations matching F on Γ with relative precision ϵ. How much can they differ at $z \in \Omega \backslash \Gamma$?

- $f = F_1 - F_2$ is small on Γ. How large is f at z relative to its global size on Ω?

Example

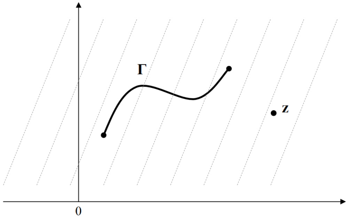


$$H^2(\mathbb{H}_+) = \{f \text{ analytic in } \mathbb{H}_+ : \sup_{y>0} \|f\|_{L^2(\mathbb{R}+iy)} < \infty\}$$

$$f(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(x)\mathrm{d}x}{x - z} = (f, p_z)_{L^2(\mathbb{R})}$$

$$p_z(\zeta) = \frac{i}{2\pi(\zeta - \overline{z})}$$

Example



$$H^2(\mathbb{H}_+) = \{f \text{ analytic in } \mathbb{H}_+ : \sup_{y>0} \|f\|_{L^2(\mathbb{R}+iy)} < \infty\}$$

$$f(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(x)\mathrm{d}x}{x-z} = (f, p_z)_{L^2(\mathbb{R})}$$

$$p_z(\zeta) = \frac{i}{2\pi(\zeta - \overline{z})}$$

Reproducing kernel Hilbert spaces (RKHS)

• $\mathcal{H}$ Hilbert space of analytic functions in Ω; $f \mapsto f(z)$ continuous $\forall z \in \Omega$, then $\exists\ p(\zeta, z) := p_z(\zeta) \in \mathcal{H}$ s.t.

$$f(z) = (f, p_z)$$

• Let $\Gamma \Subset \Omega$, introduce $\mathcal{K} : \mathcal{H} \to \mathcal{H}$

$$\mathcal{K}f(\zeta) = \int_\Gamma p(\zeta, \tau) f(\tau) |\mathrm{d}\tau|$$

• $z \in \Omega \backslash \Gamma$, $\|f\|_\Gamma := \|f\|_{L^2(\Gamma)}$

$$\begin{cases} |f(z)| \to \max \\ \|f\|_{\mathcal{H}} \le 1 \\ \|f\|_\Gamma \le \epsilon \end{cases} \qquad \longleftrightarrow \qquad \begin{cases} \Re(f, p_z) \to \max \\ (f, f) \le 1 \\ (\mathcal{K}f, f) \le \epsilon^2 \end{cases}$$

Reproducing kernel Hilbert spaces (RKHS)

- $\mathcal{H}$ Hilbert space of analytic functions in Ω; $f \mapsto f(z)$ continuous $\forall z \in \Omega$, then $\exists\ p(\zeta, z) := p_z(\zeta) \in \mathcal{H}$ s.t.

$$f(z) = (f, p_z)$$

- Let $\Gamma \Subset \Omega$, introduce $\mathcal{K} : \mathcal{H} \to \mathcal{H}$

$$\mathcal{K}f(\zeta) = \int_\Gamma p(\zeta, \tau) f(\tau) |\mathrm{d}\tau|$$

- $z \in \Omega \setminus \Gamma, \quad \|f\|_\Gamma := \|f\|_{L^2(\Gamma)}$

$$\begin{cases} |f(z)| \to \max \\ \|f\|_{\mathcal{H}} \le 1 \\ \|f\|_\Gamma \le \epsilon \end{cases} \qquad \longleftrightarrow \qquad \begin{cases} \Re(f, p_z) \to \max \\ (f, f) \le 1 \\ (\mathcal{K}f, f) \le \epsilon^2 \end{cases}$$

Reproducing kernel Hilbert spaces (RKHS)

- $\mathcal{H}$ Hilbert space of analytic functions in Ω; $f \mapsto f(z)$ continuous $\forall z \in \Omega$, then $\exists\ p(\zeta, z) := p_z(\zeta) \in \mathcal{H}$ s.t.

$$f(z) = (f, p_z)$$

- Let $\Gamma \Subset \Omega$, introduce $\mathcal{K} : \mathcal{H} \to \mathcal{H}$

$$\mathcal{K}f(\zeta) = \int_\Gamma p(\zeta, \tau) f(\tau) |\mathrm{d}\tau|$$

- $z \in \Omega \backslash \Gamma, \quad \|f\|_\Gamma := \|f\|_{L^2(\Gamma)}$

$$\begin{cases} |f(z)| \to \text{max} \\ \|f\|_{\mathcal{H}} \le 1 \\ \|f\|_\Gamma \le \epsilon \end{cases} \qquad \longleftrightarrow \qquad \begin{cases} \Re(f, p_z) \to \text{max} \\ (f, f) \le 1 \\ (\mathcal{K}f, f) \le \epsilon^2 \end{cases}$$

Theorem 1 (Quantifying stability of analytic continuation in $\mathcal{H}$)

Let $z \in \Omega\backslash\Gamma$, $f \in \mathcal{H}$ *with* $\|f\|_{\mathcal{H}} \leq 1$ *and* $\|f\|_\Gamma \leq \epsilon$, *then*

$$|f(z)| \leq \frac{3}{2} M_\epsilon(z) \tag{1}$$

where

$$M_\epsilon(\zeta) = u_\epsilon(\zeta) \min\left\{\frac{1}{\|u_\epsilon\|_{\mathcal{H}}}, \frac{\epsilon}{\|u_\epsilon\|_\Gamma}\right\}$$

and $u_\epsilon \in \mathcal{H}$ *is the unique solution of*

$$\mathcal{K}u + \epsilon^2 u = p_z$$

Theorem 1 (Quantifying stability of analytic continuation in $\mathcal{H}$)

Let $z \in \Omega \backslash \Gamma$, $f \in \mathcal{H}$ *with* $\|f\|_{\mathcal{H}} \le 1$ *and* $\|f\|_{\Gamma} \le \epsilon$, *then*

$$|f(z)| \le \frac{3}{2} M_{\epsilon}(z) \tag{1}$$

where

$$M_{\epsilon}(\zeta) = u_{\epsilon}(\zeta) \min \left\{ \frac{1}{\|u_{\epsilon}\|_{\mathcal{H}}}, \frac{\epsilon}{\|u_{\epsilon}\|_{\Gamma}} \right\}$$

and $u_{\epsilon} \in \mathcal{H}$ *is the unique solution of*

$$\mathcal{K}u + \epsilon^2 u = p_z$$

Power law

$$\mathcal{K}u + \epsilon^2 u = p_z, \qquad \mathcal{K}u(\zeta) = \int_\Gamma p(\zeta,\tau) f(\tau) |\mathsf{d}\tau|$$

- Expectation: $M_\epsilon(z) \simeq \epsilon^{\gamma(z)}$, where $\gamma = \gamma_\Gamma(z) \in (0,1)$
- Let $\{e_n\}$ be ONB of $\mathcal{H}$ with $\mathcal{K}e_n = \lambda_n e_n$

Theorem 2 (Power law)

Assume $\lambda_n \simeq e^{-\alpha n}, \quad |e_n(z)|^2 \simeq e^{-\beta n}$ *with* $0 < \beta < \alpha$. *Then, the power law principle holds with exact exponent:*

$$M_\epsilon(z) \simeq \epsilon^{\frac{\beta}{\alpha}}$$

Power law

$$\mathcal{K}u + \epsilon^2 u = p_z, \qquad \mathcal{K}u(\zeta) = \int_\Gamma p(\zeta,\tau) f(\tau) |\mathsf{d}\tau|$$

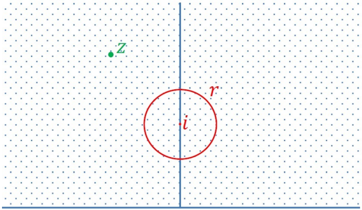
- Expectation: $M_\epsilon(z) \simeq \epsilon^{\gamma(z)}$, where $\gamma = \gamma_\Gamma(z) \in (0,1)$
- Let $\{e_n\}$ be ONB of $\mathcal{H}$ with $\mathcal{K}e_n = \lambda_n e_n$

Theorem 2 (Power law)

Assume $\lambda_n \simeq e^{-\alpha n}, \quad |e_n(z)|^2 \simeq e^{-\beta n}$ *with* $0 < \beta < \alpha$. *Then, the power law principle holds with exact exponent:*

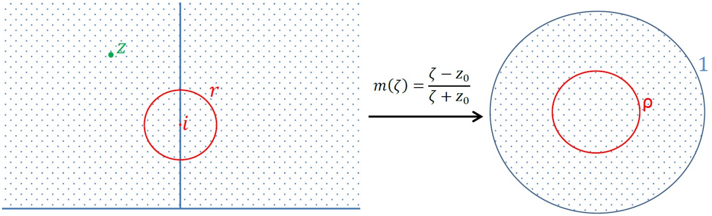
$$M_\epsilon(z) \simeq \epsilon^{\frac{\beta}{\alpha}}$$

Example $\mathcal{H} = H^2(\mathbb{H}_+),\ \Gamma = C(i,r)$



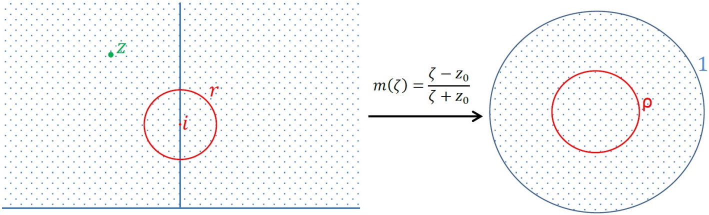
$$\lambda_n = \rho^{2n}, \qquad e_n(\zeta) = \frac{m(\zeta)^n}{\zeta + z_0} \quad \Longrightarrow \quad \gamma(z) = \frac{\ln|m(z)|}{\ln \rho}$$

Example $\mathcal{H} = H^2(\mathbb{H}_+),\ \Gamma = C(i, r)$



$$\lambda_n = \rho^{2n}, \qquad e_n(\zeta) = \frac{m(\zeta)^n}{\zeta + z_0} \qquad \Longrightarrow \qquad \gamma(z) = \frac{\ln|m(z)|}{\ln \rho}$$

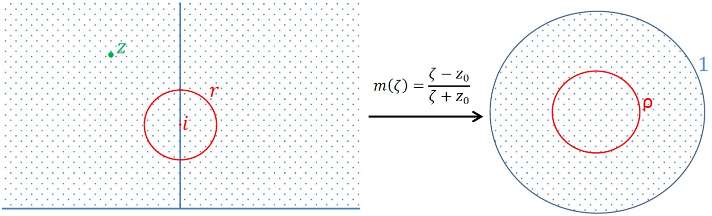
Example $\mathcal{H} = H^2(\mathbb{H}_+),\ \Gamma = C(i, r)$



$$z_0 = i\sqrt{1 - r^2}, \qquad \rho = \frac{1 - \sqrt{1 - r^2}}{r}$$

$$\lambda_n = \rho^{2n}, \qquad e_n(\zeta) = \frac{m(\zeta)^n}{\zeta + z_0} \qquad \Longrightarrow \qquad \gamma(z) = \frac{\ln |m(z)|}{\ln \rho}$$

Example $\mathcal{H} = H^2(\mathbb{H}_+),\ \Gamma = C(i, r)$



$$z_0 = i\sqrt{1 - r^2}, \qquad \rho = \frac{1 - \sqrt{1 - r^2}}{r}$$

$$\lambda_n = \rho^{2n}, \qquad e_n(\zeta) = \frac{m(\zeta)^n}{\zeta + z_0} \quad \Longrightarrow \quad \gamma(z) = \frac{\ln|m(z)|}{\ln \rho}$$

$$M(\zeta) = \frac{\epsilon^{2-\gamma(z)}}{\zeta + z_0} \sum_{n=1}^{\infty} \frac{\left(\overline{m(z)} m(\zeta)\right)^n}{\epsilon^2 + \rho^{2n}}$$

Idea of the proof: convex duality

$$\begin{cases} \Re(f, p_z) \to \text{max} \\ (f, f) - 1 \leq 0 \\ (\mathcal{K}f, f) - \epsilon^2 \leq 0 \end{cases}$$

Introduce $\mu, \nu \geq 0$, consider the Lagrangian

- target $\leq L :=$ target $- \mu$ cnstr$_1 - \nu$ cnstr$_2$
- $\nabla_f L = 0 \quad \Longrightarrow \quad f = f_{\text{opt}}(\mu, \nu)$
- subs. f_{opt} into constraints, impose equality $\Longrightarrow$ eq.'s for μ and ν

This results in

$$|f(z)| = \Re(f, p_z) \leq \frac{3}{2} u(z) \min\left\{\frac{1}{\|u\|_{\mathcal{H}}}, \frac{\epsilon}{\|u\|_{\Gamma}}\right\}$$

where $\mathcal{K}u + \epsilon^2 u = p_z$

Idea of the proof: convex duality

$$\begin{cases} \Re(f, p_z) \to \text{max} \\ (f, f) - 1 \leq 0 \\ (\mathcal{K}f, f) - \epsilon^2 \leq 0 \end{cases}$$

Introduce $\mu, \nu \geq 0$, consider the Lagrangian

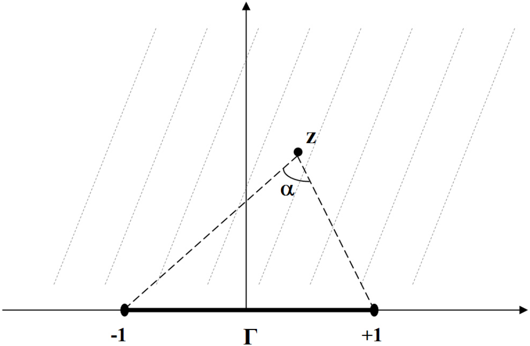
- target $\leq L :=$ target $- \mu$ cnstr$_1 - \nu$ cnstr$_2$
- $\nabla_f L = 0 \quad \Longrightarrow \quad f = f_{\text{opt}}(\mu, \nu)$
- subs. f_{opt} into constraints, impose equality $\Longrightarrow$ eq.'s for μ and ν

This results in

$$|f(z)| = \Re(f, p_z) \leq \frac{3}{2} u(z) \min \left\{ \frac{1}{\|u\|_{\mathcal{H}}}, \frac{\epsilon}{\|u\|_{\Gamma}} \right\}$$

where $\mathcal{K}u + \epsilon^2 u = p_z$

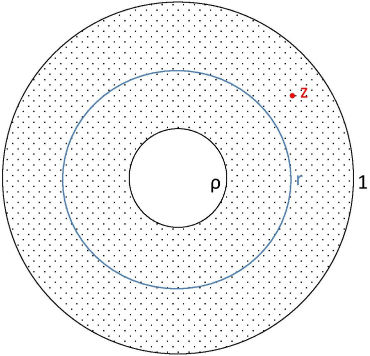
$\mathcal{H} = H^2(\mathbb{H}_+), \;\; \Gamma = [-1, 1]$



$$\gamma(z) = \frac{\alpha(z)}{\pi}$$

$$M(\zeta) = \frac{i\epsilon}{\zeta - \overline{z}} \; e^{\frac{i}{\pi} \ln \epsilon \ln \frac{1+\zeta}{1-\zeta}}$$

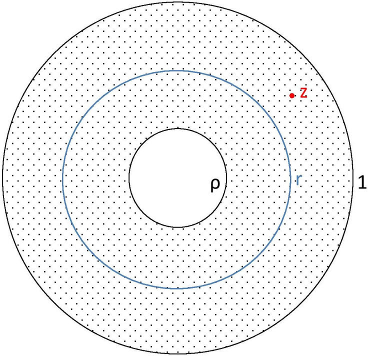
$\mathcal{H} = H^2(\{\rho < |\zeta| < 1\}), \quad \Gamma = \{|\zeta| = r\}$



$$\gamma(z) = \begin{cases} \dfrac{\ln|z|}{\ln r}, & \text{if } r < |z| < 1 \\ \dfrac{\ln(|z|/\rho)}{\ln(r/\rho)}, & \text{if } \rho < |z| < r \end{cases}$$

$$M(\zeta) = \epsilon^{2-\gamma(z)} \sum_{n \in \mathbb{Z}} \frac{(\overline{z}\zeta)^n}{r^{2n} + \epsilon^2(1+\rho^{2n})}$$

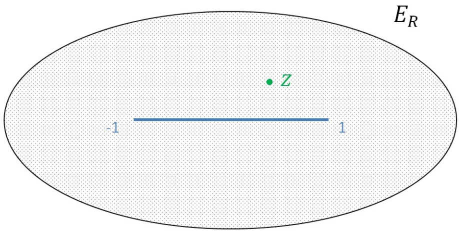
$\mathcal{H} = H^2(\{\rho < |\zeta| < 1\}), \quad \Gamma = \{|\zeta| = r\}$



$$\gamma(z) = \begin{cases} \dfrac{\ln|z|}{\ln r}, & \text{if } r < |z| < 1 \\ \dfrac{\ln(|z|/\rho)}{\ln(r/\rho)}, & \text{if } \rho < |z| < r \end{cases}$$

$$M(\zeta) = \epsilon^{2-\gamma(z)} \sum_{n\in\mathbb{Z}} \frac{(\overline{z}\zeta)^n}{r^{2n} + \epsilon^2(1+\rho^{2n})}$$

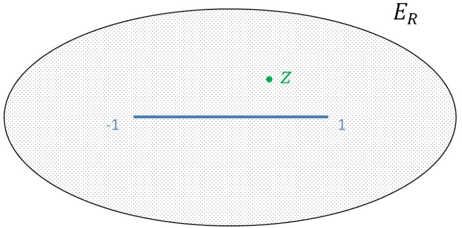
$H^\infty(E_R), \;\; \Gamma = [-1,1]$



$$\gamma(z) = 1 - \frac{\ln\left|J^{-1}(z)\right|}{\ln R}, \qquad J^{-1}(z) = z + (z-1)\sqrt{\frac{z+1}{z-1}}$$

$$M(\zeta) = \epsilon^{2-\gamma(z)} \sum_{n=1}^{\infty} \frac{(\overline{J^{-1}(z)})^n T_n(\zeta)}{1+\epsilon^2 R^{2n}}$$

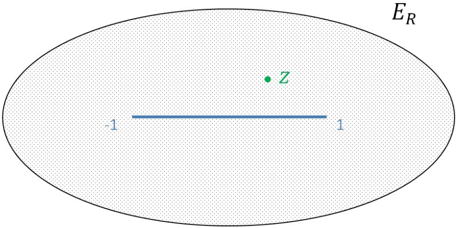
$H^\infty(E_R), \ \ \Gamma = [-1, 1]$



$$\gamma(z) = 1 - \frac{\ln\left|J^{-1}(z)\right|}{\ln R}, \qquad J^{-1}(z) = z + (z-1)\sqrt{\frac{z+1}{z-1}}$$

$$M(\zeta) = \epsilon^{2-\gamma(z)} \sum_{n=1}^{\infty} \frac{(\overline{J^{-1}(z)})^n T_n(\zeta)}{1+\epsilon^2 R^{2n}}$$

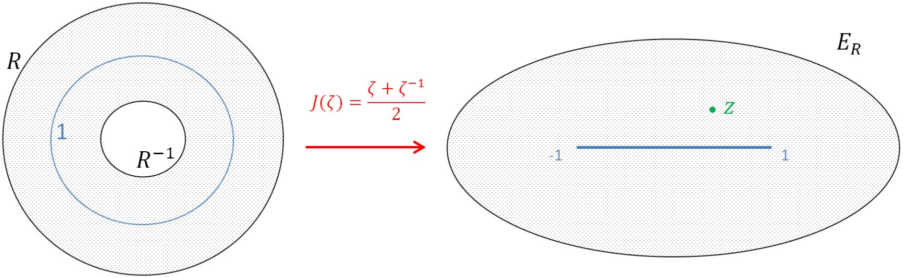
$H^\infty(E_R), \quad \Gamma = [-1, 1]$



$$\gamma(z) = 1 - \frac{\ln\left|J^{-1}(z)\right|}{\ln R}, \qquad J^{-1}(z) = z + (z-1)\sqrt{\frac{z+1}{z-1}}$$

$$M(\zeta) = \epsilon^{2-\gamma(z)} \sum_{n=1}^{\infty} \frac{(\overline{J^{-1}(z)})^n T_n(\zeta)}{1 + \epsilon^2 R^{2n}}$$

$H^\infty(E_R), \;\; \Gamma = [-1, 1]$



$$\gamma(z) = 1 - \frac{\ln\left|J^{-1}(z)\right|}{\ln R}, \qquad J^{-1}(z) = z + (z-1)\sqrt{\frac{z+1}{z-1}}$$

$$M(\zeta) = \epsilon^{2-\gamma(z)} \sum_{n=1}^{\infty} \frac{(\overline{J^{-1}(z)})^n T_n(\zeta)}{1 + \epsilon^2 R^{2n}}$$

- L. Demanet and A. Townsend
- N. Trefethen
- A. Dienstfrey and L. Greengard
- etc.