Approximate representations for the effective tensors of two-phase two-dimensional composites

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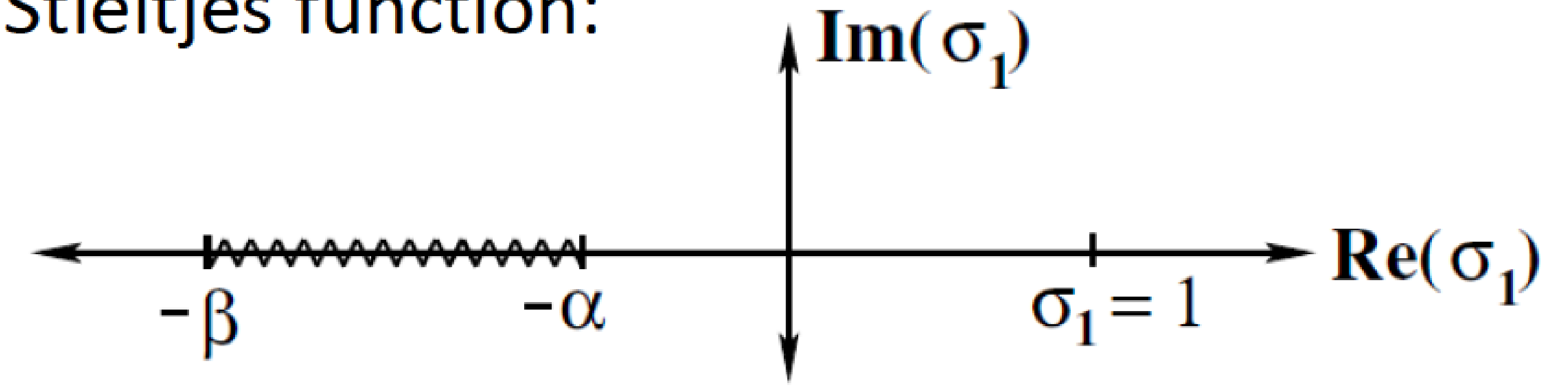
Properties of the effective conductivity

The effective conductivity σ_* is an analytic function of the component conductivities σ_1 and σ_2

With $\sigma_2 = 1$, $\sigma_*(\sigma_1)$ has the properties of a Stieltjes function:



Bergman 1978 (pioneer, but faulty arguments)

Milton 1981 (limit of resistor networks)

Golden and Papanicolaou 1983 (rigorous proof)

More generally, given Subspace Collection[Z(n)]:

$$\mathcal{H} = \mathcal{U} \oplus \mathcal{E} \oplus \mathcal{J} = \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \ldots \oplus \mathcal{P}_n$$

Projections: $\boldsymbol{\Gamma}_0$ $\boldsymbol{\Gamma}_1$ $\boldsymbol{\Gamma}_2$ $\boldsymbol{\chi}_1$ $\boldsymbol{\chi}_2$ $\boldsymbol{\chi}_n$

$$\boldsymbol{\Gamma}_0 + \boldsymbol{\Gamma}_1 + \boldsymbol{\Gamma}_2 = \mathbf{I}, \qquad \boldsymbol{\chi}_1 + \boldsymbol{\chi}_2 + \ldots + \boldsymbol{\chi}_n = \mathbf{I},$$

Pruned if $\mathcal{H}$ is the smallest subspace containing $\mathcal{U}$ that is closed under the action of $\boldsymbol{\Gamma}_1$ and $\boldsymbol{\chi}_1, \boldsymbol{\chi}_2, \cdots, \boldsymbol{\chi}_{n-1}$

Associated Herglotz function (effective tensor) defined through the abstract theory of composites

Let $\mathbf{L} = \sum_{i=1}^{n} z_i \boldsymbol{\chi}_i$

Given $\mathbf{E_0} \in \mathcal{U}$ solve

$$\mathbf{J} = \mathbf{L}\mathbf{E}, \quad \mathbf{E} \in \mathcal{U} \oplus \mathcal{E},\ \mathbf{J} \in \mathcal{U} \oplus \mathcal{J}, \quad \mathbf{E}_0 = \boldsymbol{\Gamma}_0\mathbf{E},\ \mathbf{J}_0 = \boldsymbol{\Gamma}_0\mathbf{J}$$

Since $\mathbf{J_0}$ depends linearly on $\mathbf{E_0}$:

$\mathbf{J_0} = \mathbf{L}_*\mathbf{E}_0$ defines $\mathbf{L}_*(z_1, z_2, \ldots z_n)$

$$\mathbf{L}_*(z_1, z_2, \ldots, z_n) = \boldsymbol{\Gamma}_0[(\boldsymbol{\Gamma}_0 + \boldsymbol{\Gamma}_1)\left(\sum_{i=1}^{n} \boldsymbol{\chi}_i / z_i\right)(\boldsymbol{\Gamma}_0 + \boldsymbol{\Gamma}_1)]^{-1}\boldsymbol{\Gamma}_0$$

Properties:

Homogeneity:

$$\mathbf{L}_*(\lambda z_1, \lambda z_2, \ldots, \lambda z_n) = \lambda \mathbf{L}_*(z_1, z_2, \ldots, z_n)$$

Normalization:

$$\mathbf{L}_*(1, 1, \ldots, 1) = \mathbf{I}$$

Herglotz:

$$\text{Imag}(\mathbf{L}_*) \geq 0, \quad \text{if Imag}(z_i) > 0 \text{ for all } i$$

e.g., for a composite of n isotropic phases, the $z_1, z_2, \ldots z_n$ are the conductivities $\sigma_1, \sigma_2, \ldots, \sigma_n$ of the phases and $\mathbf{L}_*$ is the effective tensor $\boldsymbol{\sigma}_*$ of the composite.

Inverse Problem: Given the function $\mathbf{L}_*(z_1, z_2, \ldots, z_n)$ can one uniquely recover the pruned subspace collection (up to an isomorphism)?

If $n = 2$. Certainly.

If $n = 3$. Maybe (open problem).

If $n \geq 4$. Certainly not.

Resolution of the open problem would be useful for finding the effective tensor for some coupled field problems e.g. thermoelectricity, given the effective conductivity function for a composite of three isotropic phases.

A representative class of geometries for two-dimensional, two-phase conducting composites having isotropic conductivities

Upshot: Any symmetric 2×2-matrix valued function satisfying the Homogeneity, Normalization, and Herglotz properties and the Keller-Dykhne-Mendelson phase interchange relationship:

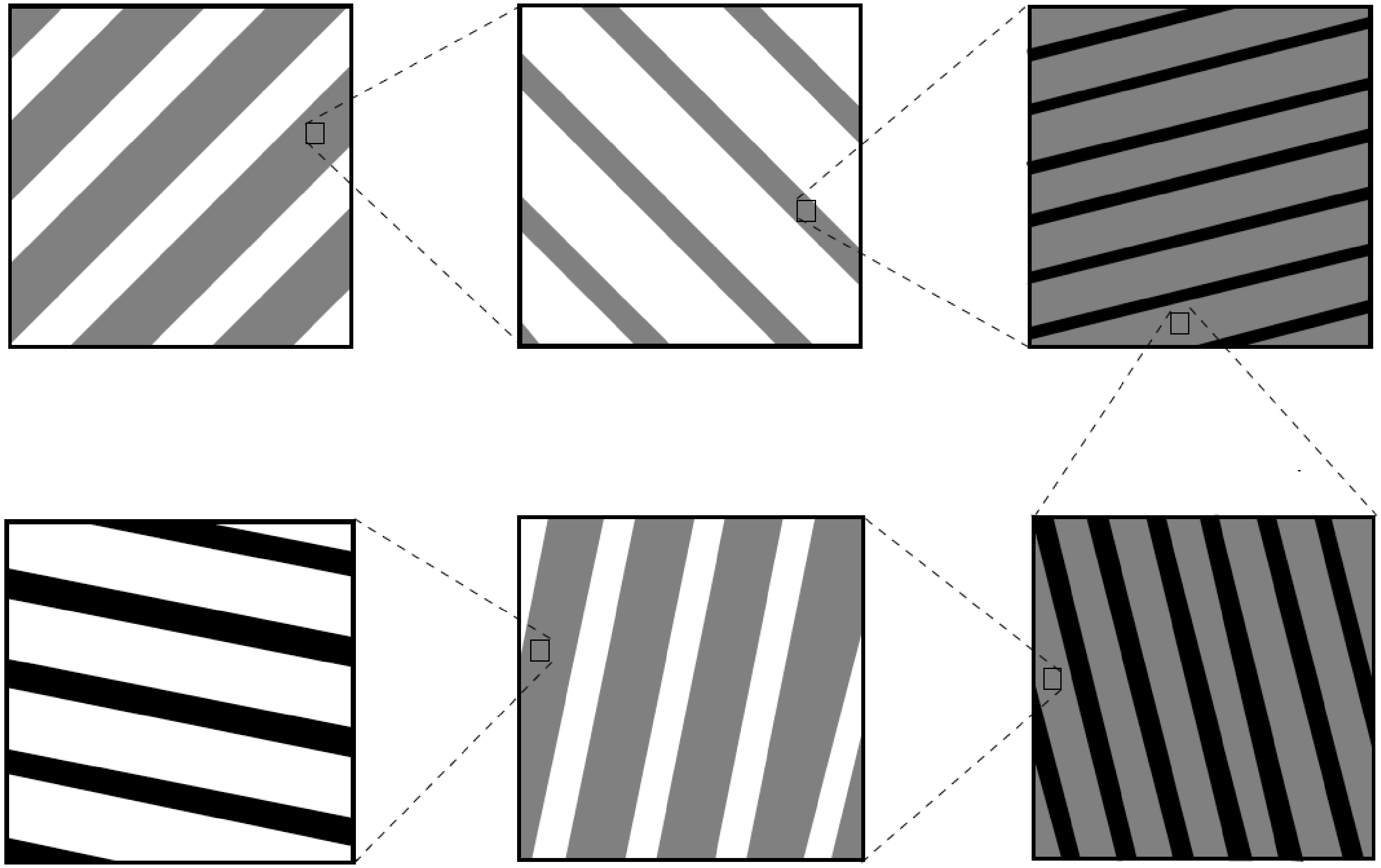
$$\boldsymbol{\sigma}^*(\sigma_2, \sigma_1) = \sigma_1 \sigma_2 \mathbf{R}_\perp [\boldsymbol{\sigma}^*(\sigma_1, \sigma_2)]^{-1} \mathbf{R}_\perp^T,$$

where $\mathbf{R}_\perp$, with transpose $\mathbf{R}_\perp^T$ is the matrix for a $90°$ rotation:

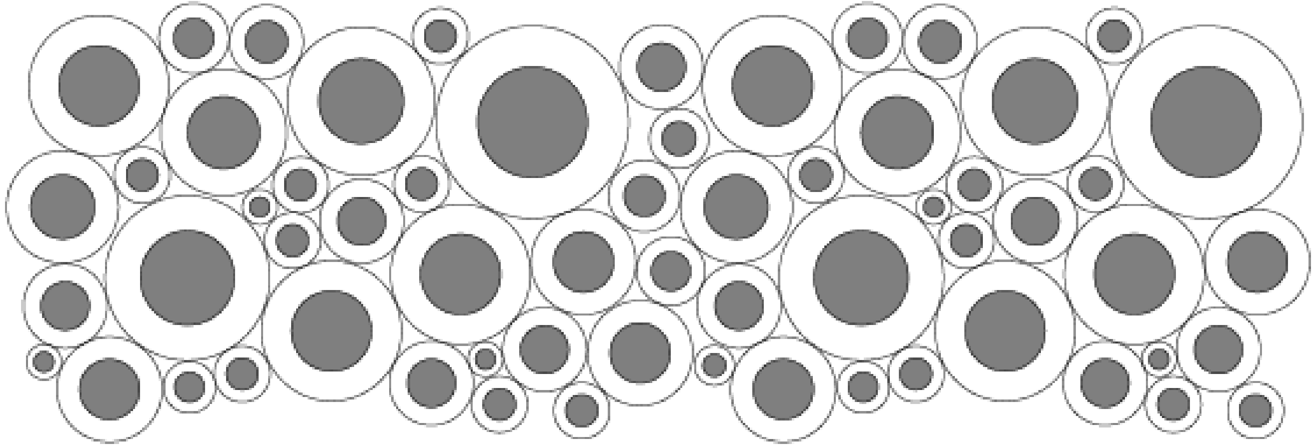
$$\mathbf{R}_\perp = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

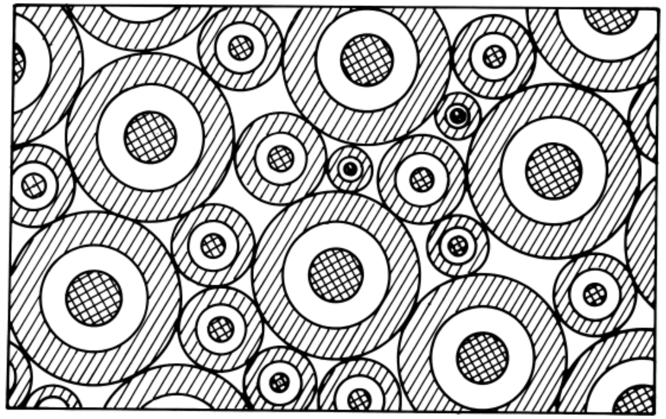
is realizable.

Hierarchical Laminates Suffice:



If the desired conductivity function is isotropic $\boldsymbol{\sigma}_*(\sigma_1, \sigma_2) = \sigma_*(\sigma_1, \sigma_2)\mathbf{I}$, then it suffices to use multicoated disk assemblages:





Key Idea

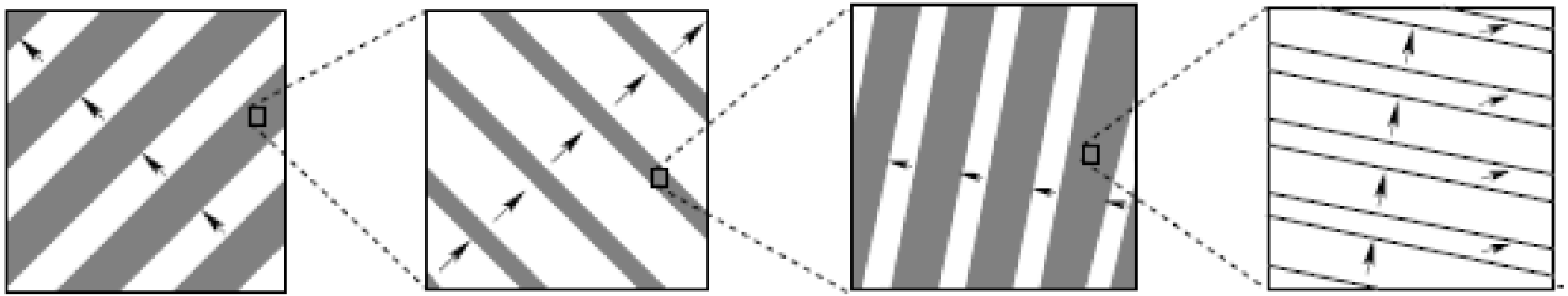
If phase 1 was the last phase used, then by setting $\sigma_2 = 0$ we insulate most of the geometry from the applied field. Thus, with $\sigma_2 = 1$ the "outermost" geometry is revealed from the residue at $\sigma_1 = \infty$ of the function $\boldsymbol{\sigma}_*(\sigma_1, 1)$.

One "peels away this layer" making the corresponding adjustment to the function. This reduces its degree. One proceeds by induction until the rational function is reduced to a constant, $\boldsymbol{\sigma}_*(\sigma_1, \sigma_2) = \sigma_1$ or σ_2.

Also a realization for the matrix valued function

$$\boldsymbol{\sigma}_*(\boldsymbol{\sigma}_0)$$

as a function of the matrix $\sigma_{\mathbf{0}}$ for two-dimensional polycrystals.
Representative structures:



with Karen Clark (1994),

Rather than looking at the effective conductivity function the idea is now to look at the associated subspace collections:

$\mathcal{U}$: space of constant vector fields.
$\mathcal{E}$: space of gradients of periodic potentials
$\mathcal{J}$: space of periodic divergence free fields that have zero mean value

Given a periodic orientation field $\mathbf{R}(\mathbf{x})$ that defines the local conductivity:

$$\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{R}(\mathbf{x})^T \boldsymbol{\sigma}_0 \mathbf{R}(\mathbf{x})$$

vectors,

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and the matrix for a 90° rotation,

$$\mathbf{R}_\perp = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

With

$$\boldsymbol{\sigma}_0 = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$$

the local conductivity takes the form

$$\boldsymbol{\sigma} = \sigma_{11}\boldsymbol{\chi}_1 + \sigma_{22}\boldsymbol{\chi}_2 + \sigma_{12}\boldsymbol{\chi}_1\mathbf{R}_\perp + \sigma_{22}\boldsymbol{\chi}_2\mathbf{R}_\perp$$

where $\boldsymbol{\chi}_1$ and $\boldsymbol{\chi}_2 = \mathbf{I} - \boldsymbol{\chi}_2$ are the projection operators:

$$\boldsymbol{\chi}_1 = \mathbf{R}^T(\mathbf{x})\mathbf{e}_1 \otimes \mathbf{e}_1\mathbf{R}(\mathbf{x}), \quad \boldsymbol{\chi}_2 = \mathbf{R}^T(\mathbf{x})\mathbf{e}_2 \otimes \mathbf{e}_2\mathbf{R}(\mathbf{x}),$$

We then have the commutation relations:

$$\mathbf{R}_\perp\boldsymbol{\chi}_1 = \boldsymbol{\chi}_2\mathbf{R}_\perp$$

$$\mathbf{R}_\perp\boldsymbol{\Gamma}_1 = \boldsymbol{\Gamma}_2\mathbf{R}_\perp$$

$$\mathbf{R}_\perp\boldsymbol{\Gamma}_0 = \boldsymbol{\Gamma}_0\mathbf{R}_\perp$$

Key Idea:

Approximate the infinite dimensional subspace collection by a finite dimensional one, and identify fields that correspond to the “last layering”.
Identify $\mathbf{v}$ and $\mathbf{v}_\perp = \mathbf{R}_\perp \mathbf{v}$ such that

$$\boldsymbol{\chi}_1 \mathbf{v} = \boldsymbol{\Gamma}_1 \mathbf{v} = 0, \quad \mathbf{v} \neq 0.$$

Counting argument: $\mathcal{E}, \mathcal{J}$ necessarily have the same dimension m. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{m+2}$ be a basis for $\mathcal{U} \oplus \mathcal{J}$ then the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{m+2}, \chi_1 \mathbf{v}_1, \chi_1 \mathbf{v}_2, \ldots, \chi_1 \mathbf{v}_{m+2}\}$ must contain exactly two linear relations:

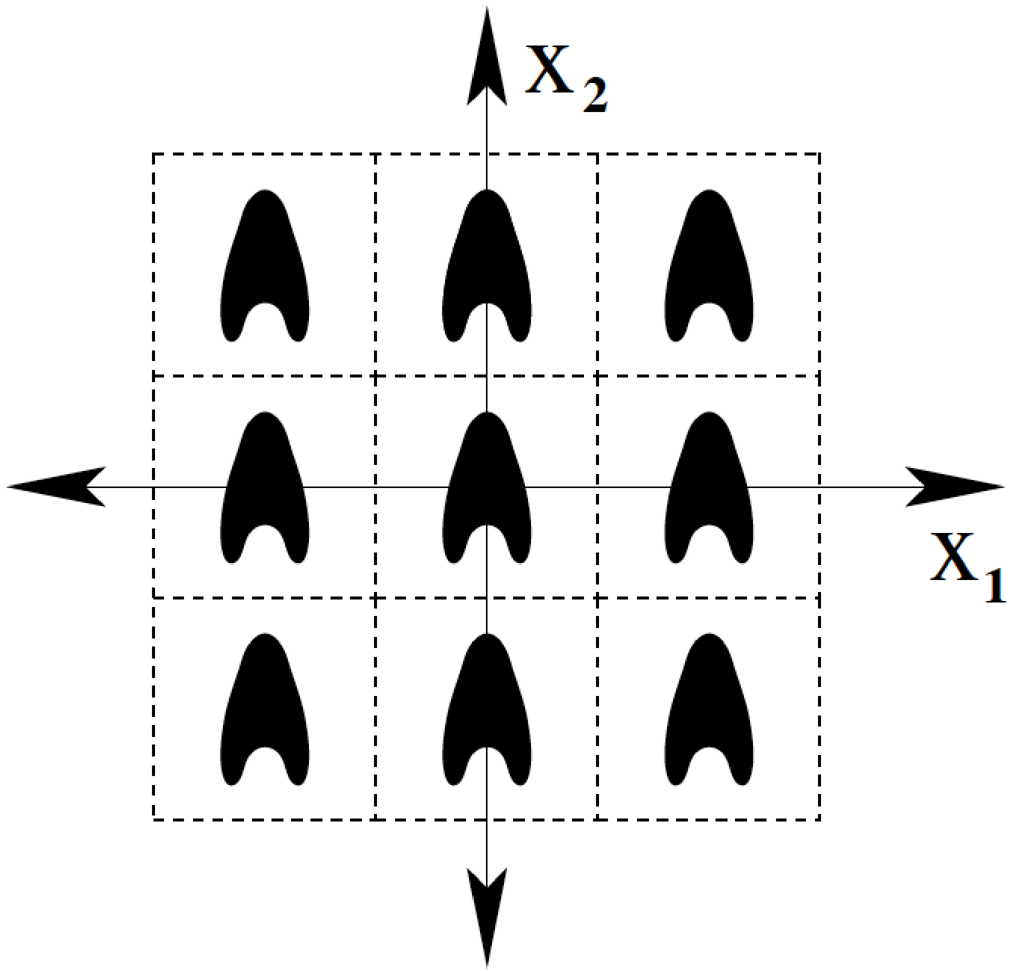
$$\sum_{i=1}^{m+2} \gamma_i \mathbf{v}_i + \mu_i \boldsymbol{\chi}_1 \mathbf{v}_i = 0, \quad \sum_{i=1}^{m+2} \gamma_i' \mathbf{v}_i + \mu_i' \boldsymbol{\chi}_1 \mathbf{v}_i = 0,$$

So we may set

$$\mathbf{v} = \sum_{i=1}^{m+2} (\gamma_i + \mu_i) \mathbf{v}_i \quad \text{or} \quad \mathbf{v} = \sum_{i=1}^{m+2} (\gamma_i' + \mu_i') \mathbf{v}_i$$

Then we may strip the fields from the subspace collection and repeat.

Representation of the effective tensor function when both phases are anisotropic, assuming a two-dimensional geometry with reflection symmetry



Starting Example:

$$\boldsymbol{\sigma}(\mathbf{x}) = \boldsymbol{\sigma}_1\chi(\mathbf{x}) + \boldsymbol{\sigma}_2(1-\chi(\mathbf{x})) \text{ with } \boldsymbol{\sigma}_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \boldsymbol{\sigma}_2 = \begin{pmatrix} \lambda_3 & 0 \\ 0 & \lambda_3 \end{pmatrix},$$

where

$$\chi(\mathbf{x}) = \begin{cases} 1 & \text{in phase 1 (the inclusions),} \\ 0 & \text{in phase 2 (the matrix).} \end{cases}$$

The assumed reflection symmetry of the geometry implies the associated effective tensor is diagonal:

$$\boldsymbol{\sigma}^* = \begin{pmatrix} \sigma^*_{11} & 0 \\ 0 & \sigma^*_{22} \end{pmatrix},$$

$\sigma^*_{11}(\lambda_1, \lambda_2, \lambda_3)$ is a Herglotz function

Theorem

Suppose the conductivity has the form (1) and consider the Domain $\mathcal{D}(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2)$ *of pairs* $(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2)$ *such that the corresponding triplet* $(\lambda_1, \lambda_2, \lambda_3)$ *satisfies*

$$c_1 \leq \mathrm{Re}(\lambda_i), \quad |\lambda_i| \leq c_2, \quad i = 1, 2, 3,$$

where c_1, c_2 *are fixed real constants with* $c_2 > c_1 > 0$. *Subject to Assumptions 1 and 2, the diagonal element* $\sigma^*_{11}(\lambda_1, \lambda_2, \lambda_3)$ *of the effective conductivity tensor* $\boldsymbol{\sigma}_*$ *can be approximated arbitrarily closely for* $(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2) \in \mathcal{D}(c_1, c_2)$ *by*

$$[\sigma^*_{11}(\lambda_1, \lambda_2, \lambda_3)]^{-1} \approx \boldsymbol{\beta} \cdot (\mathbf{Z}_2\lambda_2 + \mathbf{Z}_1\lambda_3 + \mathbf{Y}_1(\lambda_1 - \lambda_2))^{-1}\boldsymbol{\beta},$$

where $\mathbf{Z}_1, \mathbf{Z}_2 = \mathbf{I} - \mathbf{Z}_1$ *are diagonal positive definite* $\frac{m}{2} \times \frac{m}{2}$ *matrices,* $\boldsymbol{\beta}$ *is an* $m/2$*-component vector with non-negative entries, and the* $\frac{m}{2} \times \frac{m}{2}$ *matrix* $\mathbf{Y}_1$ *takes the form*

$$\mathbf{Y}_1 = \mathbf{K}^T(\mathbf{K}\mathbf{Z}_2^{-1}\mathbf{K}^T)^{-1}\mathbf{K},$$

where the $n \times \frac{m}{2}$ *matrix* $\mathbf{K}$ *has the special form*

$$\mathbf{K} = \begin{pmatrix} \mathbf{I} & \mathbf{H} \end{pmatrix}, \qquad n = \mathrm{rank}(\mathbf{Y}_1)$$

in which $\mathbf{I}$ *is the* $n \times n$ *identity matrix and* $\mathbf{H}$ *is an* $n \times (\frac{m}{2} - n)$ *matrix.*

As $\mathbf{Z}_1$ is diagonal we may write:

$$\mathbf{Z}_1 = \begin{pmatrix} \rho_1 & 0 & \cdots & 0 \\ 0 & \rho_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \rho_{m/2} \end{pmatrix}. \qquad \mathbf{Z}_2 = \mathbf{I} - \mathbf{Z}_1$$

where the ρ_i lie between 0 and 1

Assumption 1

Assume that none of the eigenvalues ρ_i of $\mathbf{Z}_1$ are 0 or 1

When $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = \lambda$ we have

$$
\begin{aligned}
[\sigma_{11}^*(1,1,\lambda)]^{-1} &= \mathbf{u}_0 \cdot \begin{pmatrix} \lambda\rho_1 + (1-\rho_1) & 0 & \cdots & 0 \\ 0 & \lambda\rho_2 + (1-\rho_2) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda\rho_m + (1-\rho_m) \end{pmatrix}^{-1} \mathbf{u}_0 \\
&= \sum_{i=1}^{m/2} \frac{\beta_i^2}{\lambda\rho_i + (1-\rho_i)},
\end{aligned}
$$

where $\mathbf{u}_0 = (\beta_1, \beta_2, \cdots, \beta_{m/2}, 0, \ldots, 0)^T$. Assuming none of the β_i are zero for $i \leq m/2$ we can determine from the poles of $[\sigma_{11}^*(1,1,\lambda)]^{-1}$ the parameters ρ_i, and hence the matrices $\mathbf{Z}_1$ and $\mathbf{Z}_2$, and from the residues we can determine the parameters β_i.

$$(\mathbf{a},\mathbf{b}) = (\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}) = \int_{\text{unit cell}} \overline{a_1(\mathbf{x})} b_1(\mathbf{x}) + \overline{a_2(\mathbf{x})} b_2(\mathbf{x}) \, d\mathbf{x},$$

$$\begin{aligned}
\mathcal{P}_1 &= \text{ all vector fields in } \mathcal{H} \text{ of the form } \begin{pmatrix} f_1(\mathbf{x}) \\ 0 \end{pmatrix}, \\
\mathcal{P}_2 &= \text{ all vector fields in } \mathcal{H} \text{ of the form } \begin{pmatrix} 0 \\ g_1(\mathbf{x}) \end{pmatrix}, \\
\mathcal{S} &= \text{ all vector fields in } \mathcal{H} \text{ of the form } \begin{pmatrix} f_2(\mathbf{x}) \\ g_2(\mathbf{x}) \end{pmatrix},
\end{aligned}$$

with periodic functions $f_1(\mathbf{x}), g_1(\mathbf{x}), f_2(\mathbf{x})$ and $g_2(\mathbf{x})$ satisfying $f_1(\mathbf{x}) \equiv g_1(\mathbf{x}) \equiv 0$ in phase 2 and $f_2(\mathbf{x}) \equiv g_2(\mathbf{x}) \equiv 0$ in phase 1.

$\mathbf{P}_1$ denote the orthogonal projection onto $\mathcal{P}_1$: $\mathbf{P}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \chi$,

$\mathbf{P}_2$ denote the orthogonal projection onto $\mathcal{P}_2$: $\mathbf{P}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \chi$,

$\mathbf{S}$ denote the orthogonal projection onto $\mathcal{S}$: $\mathbf{S} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (1 - \chi)$.

Then we have

$$\mathbf{P}_1 + \mathbf{P}_2 + \mathbf{S} = \mathbf{I}, \quad \mathbf{P}_i^T = \mathbf{P}_i, \quad \mathbf{S}^T = \mathbf{S}, \quad \mathbf{P}_i\mathbf{P}_j = \delta_{ij}\mathbf{P}_i, \quad \mathbf{P}_i\mathbf{S} = \mathbf{S}\mathbf{P}_i = 0,$$

$$\mathcal{U}_1 = \text{the one-dimensional space of fields of the form } \begin{pmatrix} e_1 \\ 0 \end{pmatrix},$$

$$\mathcal{U}_2 = \text{the one-dimensional space of fields of the form } \begin{pmatrix} 0 \\ e_2 \end{pmatrix},$$

$$\mathcal{E} = \begin{cases} \text{curl-free fields which derive from periodic potentials,} \\ \text{i.e. fields of the form } \begin{pmatrix} \frac{\partial \phi}{\partial x_1} \\ \frac{\partial \phi}{\partial x_2} \end{pmatrix} \text{ for periodic } \phi, \end{cases}$$

$$\mathcal{J} = \begin{cases} \text{divergence-free fields which derive from a periodic potentials,} \\ \text{i.e. fields of the form } \begin{pmatrix} -\frac{\partial \psi}{\partial x_2} \\ \frac{\partial \psi}{\partial x_1} \end{pmatrix} \text{ for periodic } \psi. \end{cases}$$

$\mathbf{\Lambda}_1$ denote the projection onto $\mathcal{U}_1 \oplus \mathcal{E}$,
$\mathbf{\Lambda}_2$ denote the projection onto $\mathcal{U}_2 \oplus \mathcal{J}$,

$$\mathbf{\Lambda}_1 + \mathbf{\Lambda}_2 = \mathbf{I}, \quad \mathbf{\Lambda}_i^T = \mathbf{\Lambda}_i, \quad \mathbf{\Lambda}_i \mathbf{\Lambda}_j = \delta_{ij} \mathbf{\Lambda}_i.$$

$$\mathcal{H} = \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \mathcal{S} = \mathcal{U}_1 \oplus \mathcal{U}_2 \oplus \mathcal{E} \oplus \mathcal{J},$$

Let $\mathbf{R}_\perp$ denote the operator which locally rotates the fields by 90°:

$$\mathbf{R}_\perp \begin{pmatrix} h_1(\mathbf{x}) \\ h_2(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} h_1(\mathbf{x}) \\ h_2(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} -h_2(\mathbf{x}) \\ h_1(\mathbf{x}) \end{pmatrix}.$$

Of course we have $\mathbf{R}_\perp^2 = -\mathbf{I}$ and $\mathbf{R}_\perp^T = -\mathbf{R}_\perp$. Note that

$$\mathbf{R}_\perp \mathcal{U}_1 = \mathcal{U}_2, \qquad \mathbf{R}_\perp \mathcal{E} = \mathcal{J}, \qquad \mathbf{R}_\perp \mathcal{J} = \mathcal{E},$$
$$\mathbf{R}_\perp \mathcal{P}_1 = \mathcal{P}_2, \qquad \mathbf{R}_\perp \mathcal{P}_2 = \mathcal{P}_1, \qquad \mathbf{R}_\perp \mathcal{S} = \mathcal{S},$$

or more specifically, the operators have the commutation properties

$$\begin{aligned} \mathbf{R}_\perp \mathbf{P}_1 &= \mathbf{P}_2 \mathbf{R}_\perp, \quad \mathbf{R}_\perp \mathbf{P}_2 = \mathbf{P}_1 \mathbf{R}_\perp, \quad \mathbf{R}_\perp \mathbf{S} = \mathbf{S} \mathbf{R}_\perp, \\ \mathbf{R}_\perp \mathbf{\Gamma}_0^{(1)} &= \mathbf{\Gamma}_0^{(2)} \mathbf{R}_\perp, \quad \mathbf{R}_\perp \mathbf{\Lambda}_1 = \mathbf{\Lambda}_2 \mathbf{R}_\perp, \quad \mathbf{R}_\perp \mathbf{\Lambda}_2 = \mathbf{\Lambda}_1 \mathbf{R}_\perp, \end{aligned}$$

where $\mathbf{\Gamma}_0^{(i)}$ is the projection onto $\mathcal{U}_i$ for $i = 1, 2$.

Let $\mathbf{\Pi}$ be the operator which reflects a vector field about the x_2-axis. Thus if $\mathbf{g} = \Pi\mathbf{h}$ then the two components of $\mathbf{g}$ are related to the two components of $\mathbf{h}$ via

$$g_1(x_1, x_2) = h_1(-x_1, x_2), \quad g_2(x_1, x_2) = -h_2(-x_1, x_2),$$

This operator is self-adjoint, $\mathbf{\Pi}^T = \mathbf{\Pi}$, and clearly commutes with $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{S}$, $\mathbf{\Lambda}_1$, and $\mathbf{\Lambda}_2$:

$$\mathbf{\Pi}\mathbf{P}_1 = \mathbf{P}_1\mathbf{\Pi}, \quad \mathbf{\Pi}\mathbf{P}_2 = \mathbf{P}_2\mathbf{\Pi}, \quad \mathbf{\Pi}\mathbf{S} = \mathbf{S}\mathbf{\Pi}, \quad \mathbf{\Pi}\mathbf{\Lambda}_1 = \mathbf{\Lambda}_1\mathbf{\Pi}, \quad \mathbf{\Pi}\mathbf{\Lambda}_2 = \mathbf{\Lambda}_2\mathbf{\Pi},$$

and also anticommutes with $\mathbf{R}_\perp$,

$$\mathbf{\Pi}\mathbf{R}_\perp = -\mathbf{R}_\perp\mathbf{\Pi},$$

since

$$\begin{aligned}
\mathbf{\Pi}\mathbf{R}_\perp \begin{pmatrix} h_1(x_1, x_2) \\ h_2(x_1, x_2) \end{pmatrix} &= \mathbf{\Pi} \begin{pmatrix} -h_2(x_1, x_2) \\ h_1(x_1, x_2) \end{pmatrix} = \begin{pmatrix} -h_2(-x_1, x_2) \\ -h_1(x_1, x_2) \end{pmatrix}, \\
\mathbf{R}_\perp\mathbf{\Pi} \begin{pmatrix} h_1(x_1, x_2) \\ h_2(x_1, x_2) \end{pmatrix} &= \mathbf{R}_\perp \begin{pmatrix} h_1(x_1, x_2) \\ -h_2(x_1, x_2) \end{pmatrix} = \begin{pmatrix} h_2(-x_1, x_2) \\ h_1(x_1, x_2) \end{pmatrix}.
\end{aligned}$$

Note that $\mathbf{\Pi}^2 = \mathbf{I}$ so the eigenvalues of $\mathbf{\Pi}$ are either $+1$, corresponding to eigenfunctions $\mathbf{h}^s(x_1, x_2)$ that are symmetric vector fields satisfying

$$h_1^s(x_1, x_2) = h_1^s(-x_1, x_2), \quad h_2^s(x_1, x_2) = -h_2^s(-x_1, x_2), \tag{3.23}$$

or -1, corresponding to eigenfunctions $\mathbf{h}^a(x_1, x_2)$ that are antisymmetric vector fields satisfying

$$h_1^a(x_1, x_2) = -h_1^a(-x_1, x_2) n = \mathrm{rank}(\mathbf{Y}_1)_2) = h_2^a(-x_1, x_2). \tag{3.24}$$

Accordingly, we can define

$$\begin{aligned} \mathcal{H}^s &= \text{all fields } \mathbf{h}^s \in \mathcal{H} \text{ that satisfy (3.23)}, \\ \mathcal{H}^a &= \text{all fields } \mathbf{h}^a \in \mathcal{H} \text{ that satisfy (3.24)}, \end{aligned} \tag{3.25}$$

and then $(\mathbf{I}+\mathbf{\Pi})/2$ is the projection onto $\mathcal{H}^s$, while $(\mathbf{I}-\mathbf{\Pi})/2$ is the projection onto $\mathcal{H}^a$.

Let us choose an orthonormal basis for $\mathcal{U}_1 \oplus \mathcal{E}$:

$\mathbf{u}_1,\ \mathbf{u}_2,\ \mathbf{u}_3,\ \ldots,\ \mathbf{u}_m.$

We take then the fields

$$\mathbf{v}_1 = \mathbf{R}_\perp \mathbf{u}_1,\ \mathbf{v}_2 = \mathbf{R}_\perp \mathbf{u}_2,\ \ldots,\ \mathbf{v}_m = \mathbf{R}_\perp \mathbf{u}_m,$$

as our basis for $\mathcal{U}_2 \oplus \mathcal{J}$. It follows that the $2m$ fields $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ form an orthonormal basis for $\mathcal{H}$. With respect to this basis we have

$$\mathbf{\Lambda}_1 = \begin{pmatrix} \mathbf{I} & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{\Lambda}_2 = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{I} \end{pmatrix}, \quad \mathbf{R}_\perp = \begin{pmatrix} 0 & -\mathbf{I} \\ \mathbf{I} & 0 \end{pmatrix},$$

where $\mathbf{I}$ in each case is the $m \times m$ identity matrix.

Let ρ denote an eigenvalue of the operator $\mathbf{\Lambda}_1\mathbf{S}\mathbf{\Lambda}_1$, with $\rho \neq 0$ or 1.

Let $\mathbf{e}$ be a corresponding eigenfield, $\mathbf{\Lambda}_1\mathbf{S}\mathbf{e} = \rho\mathbf{e}$. $\mathbf{e} \in \mathcal{U}_1 \oplus \mathcal{E}$.

Consider $\mathbf{e}' = \mathbf{\Lambda}_1\mathbf{R}_\perp\mathbf{S}\mathbf{e} \in \mathcal{U}_1 \oplus \mathcal{E}$.

$$
\begin{aligned}
\mathbf{\Lambda}_1\mathbf{S}\mathbf{e}' &= \mathbf{\Lambda}_1\mathbf{S}\mathbf{\Lambda}_1\mathbf{R}_\perp\mathbf{S}\mathbf{e} \\
&= \mathbf{\Lambda}_1\mathbf{S}\mathbf{R}_\perp(\mathbf{I}-\mathbf{\Lambda}_1)\mathbf{S}\mathbf{e} \quad (\text{since } \mathbf{\Lambda}_1\mathbf{R}_\perp = \mathbf{R}_\perp\mathbf{\Lambda}_2) \\
&= \mathbf{\Lambda}_1\mathbf{S}\mathbf{R}_\perp\mathbf{S}\mathbf{e} - \mathbf{\Lambda}_1\mathbf{S}\mathbf{R}_\perp\mathbf{\Lambda}_1\mathbf{S}\mathbf{e} \\
&= \mathbf{\Lambda}_1\mathbf{R}_\perp\mathbf{S}\mathbf{e} - \rho\mathbf{\Lambda}_1\mathbf{R}_\perp\mathbf{S}\mathbf{e} \quad (\text{since } \mathbf{S}\mathbf{R}_\perp = \mathbf{R}_\perp\mathbf{S} \\
&\qquad\qquad\qquad\qquad \text{and } \mathbf{\Lambda}_1\mathbf{S}\mathbf{e} = \rho\mathbf{e}) \\
&= (1-\rho)\mathbf{\Lambda}_1\mathbf{R}_\perp\mathbf{S}\mathbf{e} \\
&= (1-\rho)\mathbf{e}'.
\end{aligned}
$$

So $1-\rho$ is an eigenvalue and if $\mathbf{e}$ is a symmetric field, $\mathbf{\Pi}\mathbf{e} = \mathbf{e}$

$\mathbf{e}'$ is an antisymmetric field:

$$\mathbf{\Pi}\mathbf{e}' = \mathbf{\Pi}\mathbf{\Lambda}_1\mathbf{R}_\perp\mathbf{S}\mathbf{e} = -\mathbf{\Lambda}_1\mathbf{R}_\perp\mathbf{S}\mathbf{\Pi}\mathbf{e} = -\mathbf{\Lambda}_1\mathbf{R}_\perp\mathbf{S}\mathbf{e} = -\mathbf{e}'.$$

In an appropriate basis

$$\boldsymbol{\Lambda}_1 = \begin{pmatrix} \mathbf{I} & 0 & 0 & 0 \\ 0 & \mathbf{I} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \boldsymbol{\Pi} = \begin{pmatrix} \mathbf{I} & 0 & 0 & 0 \\ 0 & -\mathbf{I} & 0 & 0 \\ 0 & 0 & -\mathbf{I} & 0 \\ 0 & 0 & 0 & \mathbf{I} \end{pmatrix},$$

$$\boldsymbol{\Lambda}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{I} & 0 \\ 0 & 0 & 0 & \mathbf{I} \end{pmatrix},$$

$$\mathbf{R}_\perp = \begin{pmatrix} 0 & 0 & -\mathbf{I} & 0 \\ 0 & 0 & 0 & -\mathbf{I} \\ \mathbf{I} & 0 & 0 & 0 \\ 0 & \mathbf{I} & 0 & 0 \end{pmatrix}. \qquad \mathbf{U}_1 = \begin{pmatrix} \boldsymbol{\beta} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{\beta} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{m/2} \end{pmatrix},$$

$$\mathbf{S} = \begin{pmatrix} \mathbf{Z}_1 & 0 & 0 & -(\mathbf{Z}_1\mathbf{Z}_2)^{1/2} \\ 0 & \mathbf{Z}_2 & (\mathbf{Z}_1\mathbf{Z}_2)^{1/2} & 0 \\ 0 & (\mathbf{Z}_1\mathbf{Z}_2)^{1/2} & \mathbf{Z}_1 & 0 \\ -(\mathbf{Z}_1\mathbf{Z}_2)^{1/2} & 0 & 0 & \mathbf{Z}_2 \end{pmatrix},$$

$$\mathbf{P}_1 = \left(\begin{array}{cc:cc} \mathbf{Y}_1 & 0 & 0 & \mathbf{Y}_1\mathbf{Q} \\ 0 & \mathbf{Y}_2 & -\mathbf{Y}_2\mathbf{Q}^{-1} & 0 \\ \hdashline 0 & -\mathbf{Q}^{-1}\mathbf{Y}_2 & \mathbf{Q}^{-1}\mathbf{Y}_2\mathbf{Q}^{-1} & 0 \\ \mathbf{Q}\mathbf{Y}_1 & 0 & 0 & \mathbf{Q}\mathbf{Y}_1\mathbf{Q} \end{array}\right), \qquad \mathbf{Q} = \sqrt{\mathbf{Z}_1\mathbf{Z}_2^{-1}}$$

$$\mathbf{P}_2 = \begin{pmatrix} \mathbf{Q}^{-1}\mathbf{Y}_2\mathbf{Q}^{-1} & 0 & 0 & \mathbf{Q}^{-1}\mathbf{Y}_2 \\ 0 & \mathbf{Q}\mathbf{Y}_1\mathbf{Q} & -\mathbf{Q}\mathbf{Y}_1 & 0 \\ 0 & -\mathbf{Y}_1\mathbf{Q} & \mathbf{Y}_1 & 0 \\ \mathbf{Y}_2\mathbf{Q}^{-1} & 0 & 0 & \mathbf{Y}_2 \end{pmatrix}.$$

We require the technical

Assumption 2

We assume the fields

$$\mathbf{w}_1 = \Upsilon_1 \mathbf{u}_1, \quad \mathbf{w}_2 = \Upsilon_1 \mathbf{u}_2, \quad \ldots, \quad \mathbf{w}_n = \Upsilon_1 \mathbf{u}_n, \tag{3.75}$$

are non-zero and independent, where Υ_1 is the projection onto the range of $\mathbf{Y}_1$ and the $\mathbf{u}_i$ are orthonormal eigenfields of $\mathbf{\Lambda}_1 \mathbf{S} \mathbf{\Lambda}_1$

Thus $\Upsilon_1 \mathbf{u}_i = \sum_{a=1}^{n} \mathbf{w}_a K_{ai}, \;\; \mathbf{K} = (\mathbf{I} \;\; \mathbf{H})$,

and after some algebra we get

$$\mathbf{Y}_1 = \mathbf{K}^T (\mathbf{K} \mathbf{Z}_2^{-1} \mathbf{K}^T)^{-1} \mathbf{K}, \quad \mathbf{Y}_2 = \mathbf{Z}_1 - \mathbf{Q} \mathbf{Y}_1 \mathbf{Q}, \quad \mathbf{Q} = (\mathbf{Z}_1 \mathbf{Z}_2^{-1})^{1/2} \text{ and } \mathbf{Z}_2 = \mathbf{I} - \mathbf{Z}_1$$

Having obtained representation formulas for the relevant operators one just needs to substitute them in the formula for the effective tensor.

The case where both phases are anisotropic.

$$
\begin{aligned}
\mathcal{P}_1 &= \text{all vector fields of the form } \begin{pmatrix} f_1(\mathbf{x}) \\ 0 \end{pmatrix}, \\
\mathcal{P}_2 &= \text{all vector fields of the form } \begin{pmatrix} 0 \\ g_1(\mathbf{x}) \end{pmatrix}, \\
\mathcal{P}_3 &= \text{all vector fields of the form } \begin{pmatrix} f_2(\mathbf{x}) \\ 0 \end{pmatrix}, \\
\mathcal{P}_4 &= \text{all vector fields of the form } \begin{pmatrix} 0 \\ g_2(\mathbf{x}) \end{pmatrix},
\end{aligned}
$$

with periodic functions $f_1(\mathbf{x})$, $g_1(\mathbf{x})$, $f_2(\mathbf{x})$ and $g_2(\mathbf{x})$ satisfying $f_1(\mathbf{x}) \equiv g_1(\mathbf{x}) \equiv 0$ in phase 2 and $f_2(\mathbf{x}) \equiv g_2(\mathbf{x}) \equiv 0$ in phase 1.

$\mathbf{P}_1$ denote the orthogonal projection onto $\mathcal{P}_1 : \mathbf{P}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \chi,$

$\mathbf{P}_2$ denote the orthogonal projection onto $\mathcal{P}_2 : \mathbf{P}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \chi,$

$\mathbf{P}_3$ denote the orthogonal projection onto $\mathcal{P}_3 : \mathbf{P}_3 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} (1 - \chi),$

$\mathbf{P}_4$ denote the orthogonal projection onto $\mathcal{P}_4 : \mathbf{P}_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} (1 - \chi).$

$$\mathbf{P}_1 = \begin{pmatrix} \mathbf{Y}_1 & 0 & 0 & \mathbf{Y}_1\mathbf{Q} \\ 0 & \mathbf{Y}_2 & -\mathbf{Y}_2\mathbf{Q}^{-1} & 0 \\ 0 & -\mathbf{Q}^{-1}\mathbf{Y}_2 & \mathbf{Q}^{-1}\mathbf{Y}_2\mathbf{Q}^{-1} & 0 \\ \mathbf{Q}\mathbf{Y}_1 & 0 & 0 & \mathbf{Q}\mathbf{Y}_1\mathbf{Q} \end{pmatrix}, \qquad \mathbf{Q} = \mathbf{Z}_1^{1/2}\mathbf{Z}_2^{-1/2}, \quad \mathbf{Z}_2 = \mathbf{I} - \mathbf{Z}_1,$$

$$\mathbf{P}_2 = \begin{pmatrix} \mathbf{Q}^{-1}\mathbf{Y}_2\mathbf{Q}^{-1} & 0 & 0 & \mathbf{Q}^{-1}\mathbf{Y}_2 \\ 0 & \mathbf{Q}\mathbf{Y}_2\mathbf{Q} & -\mathbf{Q}\mathbf{Y}_1 & 0 \\ 0 & -\mathbf{Y}_1\mathbf{Q} & \mathbf{Y}_1 & 0 \\ \mathbf{Y}_2\mathbf{Q}^{-1} & 0 & 0 & \mathbf{Y}_2 \end{pmatrix}, \qquad \mathbf{Z}_1 = \begin{pmatrix} \rho_1 & 0 & \cdots & 0 \\ 0 & \rho_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \rho_{m/2} \end{pmatrix}$$

$$\mathbf{P}_3 = \begin{pmatrix} \mathbf{Y}_3 & 0 & 0 & \mathbf{Y}_3\mathbf{Q}^{-1} \\ 0 & \mathbf{Y}_4 & -\mathbf{Y}_4\mathbf{Q} & 0 \\ 0 & -\mathbf{Q}\mathbf{Y}_4 & \mathbf{Q}\mathbf{Y}_4\mathbf{Q} & 0 \\ \mathbf{Q}^{-1}\mathbf{Y}_3 & 0 & 0 & \mathbf{Q}^{-1}\mathbf{Y}_3\mathbf{Q}^{-1} \end{pmatrix},$$

$$\mathbf{P}_4 = \begin{pmatrix} \mathbf{Q}\mathbf{Y}_4\mathbf{Q} & 0 & 0 & \mathbf{Q}\mathbf{Y}_4 \\ 0 & \mathbf{Q}^{-1}\mathbf{Y}_3\mathbf{Q}^{-1} & -\mathbf{Q}^{-1}\mathbf{Y}_3 & 0 \\ 0 & -\mathbf{Y}_3\mathbf{Q}^{-1} & \mathbf{Y}_3 & 0 \\ \mathbf{Y}_4\mathbf{Q} & 0 & 0 & \mathbf{Y}_4 \end{pmatrix}. \qquad \mathbf{R}_\perp = \begin{pmatrix} 0 & 0 & -\mathbf{I} & 0 \\ 0 & 0 & 0 & -\mathbf{I} \\ \mathbf{I} & 0 & 0 & 0 \\ 0 & \mathbf{I} & 0 & 0 \end{pmatrix}.$$

$$\mathbf{Y}_1 = \mathbf{K}_1^T(\mathbf{K}_1\mathbf{Z}_2^{-1}\mathbf{K}_1^T)^{-1}\mathbf{K}_1, \qquad \mathbf{Y}_2 = \mathbf{Z}_1 - \mathbf{Q}\mathbf{Y}_1\mathbf{Q},$$
$$\mathbf{Y}_3 = \mathbf{K}_2^T(\mathbf{K}_2\mathbf{Z}_1^{-1}\mathbf{K}_2^T)^{-1}\mathbf{K}_2 \qquad \mathbf{Y}_4 = \mathbf{Z}_2 - \mathbf{Q}^{-1}\mathbf{Y}_3\mathbf{Q}^{-1},$$

$\mathbf{K}_1 = (\mathbf{I} \quad \mathbf{H}_1)$: $\mathbf{I}$ is the $n_1 \times n_1$ identity, $\mathbf{H}_1$ is $n_1 \times (\frac{m}{2} - n_1)$,
$\mathbf{K}_2 = (\mathbf{I} \quad \mathbf{H}_2)$: $\mathbf{I}$ is the $n_2 \times n_2$ identity, $\mathbf{H}_2$ is $n_2 \times (\frac{m}{2} - n_2)$.

$$n_1 = \text{rank}\,\mathbf{Y}_1, \quad n_2 = \text{rank}\,\mathbf{Y}_3$$

$$\mathbf{H}_1^T \begin{pmatrix} (1-\rho_1)\beta_1 \\ (1-\rho_2)\beta_2 \\ \vdots \\ (1-\rho_{n_1})\beta_{n_1} \end{pmatrix} = \begin{pmatrix} (1-\rho_{n_1+1})\beta_{n_1+1} \\ (1-\rho_{n_1+2})\beta_{n_1+2} \\ \vdots \\ (1-\rho_{m/2})\beta_{m/2} \end{pmatrix}. \qquad \mathbf{H}_2^T \begin{pmatrix} \rho_1\beta_1 \\ \rho_2\beta_2 \\ \vdots \\ \rho_{n_2}\beta_{n_2} \end{pmatrix} = \begin{pmatrix} \rho_{n_2+1}\beta_{n_2+1} \\ \rho_{n_2+2}\beta_{n_2+2} \\ \vdots \\ \rho_{m/2}\beta_{m/2} \end{pmatrix}.$$

$$
\begin{aligned}
\boldsymbol{\sigma}(\mathbf{x}) &= \boldsymbol{\sigma}_1\chi + \boldsymbol{\sigma}_2(1-\chi) = \begin{pmatrix} \sigma_{1,11} & \sigma_{1,12} \\ \sigma_{1,21} & \sigma_{1,22} \end{pmatrix} \chi + \begin{pmatrix} \sigma_{2,11} & \sigma_{2,12} \\ \sigma_{2,21} & \sigma_{2,22} \end{pmatrix} (1-\chi) \\
&= \begin{pmatrix} \sigma_{1,11} & 0 \\ 0 & 0 \end{pmatrix} \chi + \begin{pmatrix} 0 & 0 \\ 0 & \sigma_{1,22} \end{pmatrix} \chi \\
&\quad + \begin{pmatrix} \sigma_{1,12} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \chi - \begin{pmatrix} 0 & 0 \\ 0 & \sigma_{1,21} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \chi \\
&\quad + \begin{pmatrix} \sigma_{2,11} & 0 \\ 0 & 0 \end{pmatrix} (1-\chi) + \begin{pmatrix} 0 & 0 \\ 0 & \sigma_{2,22} \end{pmatrix} (1-\chi) \\
&\quad + \begin{pmatrix} \sigma_{2,12} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (1-\chi) \\
&\quad - \begin{pmatrix} 0 & 0 \\ 0 & \sigma_{2,21} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (1-\chi) \\
&= \sigma_{1,11}\mathbf{P}_1 + \sigma_{1,22}\mathbf{P}_2 + \sigma_{1,12}\mathbf{P}_1\mathbf{R}_\perp - \sigma_{1,21}\mathbf{P}_2\mathbf{R}_\perp \\
&\quad + \sigma_{2,11}\mathbf{P}_3 + \sigma_{2,22}\mathbf{P}_4 + \sigma_{2,12}\mathbf{P}_3\mathbf{R}_\perp - \sigma_{2,21}\mathbf{P}_4\mathbf{R}_\perp .
\end{aligned}
$$

More generally, for elasticity and other coupled equations

$$\begin{pmatrix} \mathbf{j}^{(1)}(\mathbf{x}) \\ \mathbf{j}^{(2)}(\mathbf{x}) \\ \vdots \\ \mathbf{j}^{(k)}(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} \boldsymbol{\sigma}^{(11)}(\mathbf{x}) & \boldsymbol{\sigma}^{(12)}(\mathbf{x}) & \dots & \boldsymbol{\sigma}^{(1k)}(\mathbf{x}) \\ \boldsymbol{\sigma}^{(21)}(\mathbf{x}) & \boldsymbol{\sigma}^{(22)}(\mathbf{x}) & \dots & \boldsymbol{\sigma}^{(2k)}(\mathbf{x}) \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{\sigma}^{(k1)}(\mathbf{x}) & \boldsymbol{\sigma}^{(k2)}(\mathbf{x}) & \dots & \boldsymbol{\sigma}^{(kk)}(\mathbf{x}) \end{pmatrix} \begin{pmatrix} \mathbf{e}^{(1)}(\mathbf{x}) \\ \mathbf{e}^{(2)}(\mathbf{x}) \\ \vdots \\ \mathbf{e}^{(k)}(\mathbf{x}) \end{pmatrix}$$

$$\nabla \cdot \mathbf{j}^{(i)} = 0, \quad \mathbf{e}^{(j)} = \nabla V_j, \quad \boldsymbol{\sigma}^{(ij)}(\mathbf{x}) = \chi(\mathbf{x})\boldsymbol{\sigma}_1^{(ij)} + [1 - \chi(\mathbf{x})]\boldsymbol{\sigma}_2^{(ij)},$$

$$\begin{aligned} \boldsymbol{\sigma}^{(ij)} \;=\; & \sigma_{1,11}^{(ij)}\mathbf{P}_1 + \sigma_{1,22}^{(ij)}\mathbf{P}_2 + \sigma_{1,12}^{(ij)}\mathbf{P}_1\mathbf{R}_\perp - \sigma_{1,21}^{(ij)}\mathbf{P}_2\mathbf{R}_\perp \\ & + \sigma_{2,11}^{(ij)}\mathbf{P}_3 + \sigma_{2,22}^{(ij)}\mathbf{P}_4 + \sigma_{2,12}^{(ij)}\mathbf{P}_3\mathbf{R}_\perp - \sigma_{2,21}^{(ij)}\mathbf{P}_4\mathbf{R}_\perp. \end{aligned}$$

Question (open):
Are laminates, and laminates of laminates
a representative class of structures?

Thank You!

Reference:

GWM. Approximating the effective tensor as a function of the component tensors in two-dimensional composites of two anisotropic phases,
SIAM Journal on Mathematical Analysis 50(3), 3327--3364,
DOI: 10.1137/17M1130356 (2018)

and references therein: