Integral representation formulas (IRF) for permeability and tortuosity for porous media

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Outline

- Governing equations of wave propagation in poroelastic materials
- Memory terms and the visco-dynamics
- Permeability and tortuosity
- Stieltjes function representation of Johnson-Koplik-Dashen (JKD) permeability and tortuosity
- Padé approximation and two-sided approximation with built-in asymptotic behaviors
- Preliminary results on Integral representation of stationary permeability

Wave propagation in poroelastic materials

M. A. Biot, *Theory of propagation of elastic waves in a fluid-saturated porous solid*, J. Acoustical Society of America (JASA), 1956. Two papers - one for low frequency and the other for high frequency.

$\boldsymbol{u}$: displacement vector for solid
$\boldsymbol{U}$: displacement vector for fluid
ϕ: porosity
p: pore fluid pressure, $\boldsymbol{v} := \dot{\boldsymbol{u}}, \quad \boldsymbol{q} := \phi(\dot{\boldsymbol{U}} - \dot{\boldsymbol{u}})$

- Equations of motion

$$\rho \partial_t v_j + \rho_f \partial_t q_j = [\nabla \cdot \tau]_j, \; j = 1, 2, 3$$

- Generalized Darcy's Law

$$-\frac{\partial p}{\partial x_j} = \rho_f \frac{\partial v_j}{\partial t} + \left(\frac{\rho_f}{\phi}\right) \check{\alpha}_j \star \frac{\partial q_j}{\partial t}, \; j = 1, 2, 3$$

Dynamic permeability and tortuosity

Dynamic permeability function $K(\omega)$

$$-i\omega\phi(\hat{\boldsymbol{U}} - \hat{\boldsymbol{u}}) = \frac{K(\omega)}{\eta}(-\nabla\hat{p} + \rho_f\omega^2\hat{\boldsymbol{u}}) \tag{1}$$

Dynamic tortuosity function $\alpha(\omega)$

$$\alpha(\omega)\rho_f(-i\omega)^2(\hat{\boldsymbol{U}} - \hat{\boldsymbol{u}}) = (-\nabla\hat{p} + \rho_f\omega^2\hat{\boldsymbol{u}}), \tag{2}$$

$$(2) \text{ and } (1) \Rightarrow \alpha(\omega) = \frac{i\eta\phi}{\omega\rho_f}K^{-1}(\omega) \text{ for } \omega \neq 0$$

Note: $K(0) = K_0$(static permeability)

Memory term

$$-\frac{\partial p}{\partial x_j} = \rho_f \frac{\partial v_j}{\partial t} + \left(\frac{\rho_f}{\phi}\right) \check{\alpha}_j \star \frac{\partial q_j}{\partial t}, \; j = 1, 2, 3$$

Note: For the special case of low frequency Biot's equation,

$$-\partial_{x_i} p = \rho_f \partial_t v_i + \left(\frac{\rho_f}{\phi}\right) \alpha_{\infty i} \partial_t q_i + \left(\frac{\eta}{K_{0i}}\right) q_i$$

i.e. $\check{\alpha}_j(t) = \alpha_{\infty j}\delta(t) + \frac{\eta\phi}{K_{\mathbf{0}j}\rho_f} H(t) \Longleftrightarrow \alpha_j(\omega) = \alpha_{\infty j} + \frac{\eta\phi/(K_{\mathbf{0}j}\rho_f)}{-i\omega}$

$\alpha_{\infty i}$: inf-freq tortuosity in the i-th direction

Transform to frequency domain

$$\hat{f}(\omega) := \mathcal{L}[f](s = -i\omega) := \int_0^\infty f(t) e^{-st} dt$$

JKD Model

In [Johnson-Koplik-Dashen-1987], by extending $\alpha(\omega)$ and $K(\omega)$ to complex ω-plane and using causality argument, the simplest forms are derived

$$\alpha_{Dj}(\omega) = \alpha_{\infty j}\left(1 - \frac{\eta\phi}{i\omega\alpha_{\infty j}\rho_f K_{0j}}\sqrt{1 - i\frac{4\alpha_{\infty j}^2 K_{0j}^2\rho_f\omega}{\eta\Lambda_j^2\phi^2}}\right)$$

$$K_{Dj}(\omega) = K_{0j}/\left(\sqrt{1 - \frac{4i\alpha_{\infty j}^2 K_{0j}^2\rho_f\omega}{\eta\Lambda_j^2\phi^2}} - \frac{i\alpha_{\infty j}K_{0j}\rho_f\omega}{\eta\phi}\right).$$

with the tunable geometry-dependent constant Λ_j. inf-freq. model as $\omega \to \infty$

$$\alpha(\omega) \to \alpha_\infty\left(1 + \sqrt{\frac{i\eta}{\rho_f\omega}}\frac{2}{\Lambda}\right), \; K(\omega) \to \frac{i\eta\phi}{\alpha_\infty\rho_f\omega}\left(1 - \sqrt{\frac{i\eta}{\rho_f\omega}}\frac{2}{\Lambda}\right)$$

IRF for JKD dynamic permeability

Theorem ([Ou-2014])

The JKD permeability can be represented as

$$K^D(\omega) = \frac{\nu}{F} \int_0^{\xi_p} \frac{u dG(u)}{1 - i\omega u}$$

where the probability measure dG is

$$dG(u) = \chi_{\mathcal{I}}(u) \left(\frac{\psi(u)}{u} \right) du + \left(\frac{r}{\xi_p} \right) \delta_{\xi_p},$$

IRF for Dynamic Tortuosity

Theorem (Ou-2014)

The dynamic tortuosity $\alpha(\omega) = \frac{i\eta\phi}{\omega\rho_f}K^{-1}$ has the following integral representation formula for ω such that $-\frac{i}{\omega} \in \mathbb{C} \setminus [0, \Theta_1]$

$$\alpha(\omega) = a\left(\frac{i}{\omega}\right) + \int_0^{\Theta_1} \frac{d\sigma(\Theta)}{1 - i\omega\Theta}$$

for some positive measure $d\sigma$, with $a = \frac{\eta\phi}{\rho_f K_0}$.

(1) $\alpha(\omega) \to \alpha_\infty$ as $\omega \to \infty$, $d\sigma$ has a Dirac mass at $\Theta = 0$ with strength α_∞.

(2) $\alpha(\omega) \approx \frac{a}{-i\omega} + \alpha_\infty + \sum_{j=1}^{M} \frac{r_j}{-i\omega - p_j}$, $r_j > 0$, $p_j \in (-\infty, -\frac{1}{\Theta_1})$

M.Y. Ou, On reconstruction of dynamic permeability and tortuosity from data at distinct frequencies, Inverse Problems 30(9) 095002, 2014

Wave equations with no explicit memory term [Ou-Woerdeman-2019]

$$\Theta_k^{x_j}(\boldsymbol{x}, t) := (-p_k)e^{p_k t}$$

$$\begin{cases} \sum_{k=1}^{3} \frac{\partial \tau_{jk}}{\partial x_k} = \rho \frac{\partial v_j}{\partial t} + \rho_f \frac{\partial q_j}{\partial t}, \ t > 0, & (3) \\ \partial_t \Theta_k^{x_j}(\boldsymbol{x}, t) = p_k^{x_j} \Theta_k^{x_j}(\boldsymbol{x}, t) - p_k^{x_j} q_j(\boldsymbol{x}, t), \ j = 1, 2, 3, & (4) \\ -\frac{\partial p}{\partial x_j} = \rho_f \frac{\partial v_j}{\partial t} + \left(\frac{\rho_f \alpha_{\infty j}}{\phi} \right) \frac{\partial q_j}{\partial t} + \left(\frac{\eta}{\kappa_j} + \frac{\rho_f}{\phi} \sum_{k=1}^{M} r_k^{x_j} \right) q_j \\ \quad - \left(\frac{\rho_f}{\phi} \right) \sum_{k=1}^{M_j} r_k^{x_j} \Theta_k^{x_j}, \ t > 0, \ j = 1, 2, 3, & (5) \end{cases}$$

Reconstruction of $\alpha(\omega)$: Rational function approximation [Ou-Woerdeman-2019]

$s_k := -i\omega_k,\ k = 1, \cdots, N$ be the interpolation points with $Im(s_k) \neq 0$

$$\begin{aligned} D(s_k) - \alpha_\infty &= \int_0^\theta \frac{d\sigma(t)}{1 + s_k t} \approx [N-1/N]_{D(s)-\alpha_\infty} \\ &:= \frac{a_0 + a_1 s_k + \cdots + a_{N-1} s_k^{N-1}}{1 + b_1 s_k + \cdots + b_N s_k^N},\ k = 1, 2, \cdots, N. \end{aligned} \tag{6}$$

$D(\overline{s_k}) = \overline{D(s_k)}$

$$\begin{cases} D(s_k) - \alpha_\infty = \dfrac{a_0 + a_1 s_k + \cdots + a_{N-1} s_k^{N-1}}{1 + b_1 s_k + \cdots + b_N s_k^N}, k = 1, \ldots, N, & (7) \\ \overline{D(s_k)} - \alpha_\infty = \dfrac{a_0 + a_1 \overline{s_k} + \cdots + a_{N-1} \overline{s_k}^{N-1}}{1 + b_1 \overline{s_k} + \cdots + b_{N-1} \overline{s_k}^N}, k = 1, \ldots, N. & (8) \end{cases}$$

JK Gelfgren, Multipoint Padé approximants converging to functions of Stieltjes type. In Padé Approximation and its Applications Amsterdam, pp.197-207. Springer, 1981

Method based on two-sided residue interpolation

$$
\begin{aligned}
u_k &= \frac{1}{s_k}, \; v_k = D\left(s_k\right) - \alpha_\infty, k = 1 \dots N \\
\left(S_1\right)_{pq} &= \frac{-s_q D\left(s_q\right) + s_p^* D^*\left(s_p\right)}{s_p^* - s_q} - \alpha_\infty, \; p, \; q = 1 \dots N, \\
\left(S_2\right)_{pq} &= \frac{-D\left(s_q\right) + D^*\left(s_p\right)}{s_q - s_p^*}, \; p, \; q = 1 \dots N, \\
C_- := & \quad \left(u_1, \cdots, \; u_N\right), \; C_+ := \left(v_1, \cdots, v_N\right)
\end{aligned}
$$

$$\mathbf{S}_1\mathbf{V} = \mathbf{S}_2\mathbf{V}\Phi$$

$$
\begin{aligned}
p_k &= -\Phi(k, k) \\
r_k &= C_+\mathbf{V}(:, k)\mathbf{V}(:, k)^* C_+^*
\end{aligned}
$$

D. Alpay, J. A. Ball, I. Gohberg, and L. Rodman. The two-sided residue interpolation in the Stieltjes class for matrix functions, Linear Algebra and its Applications, 208/209:485–521, 1994

Pole-residue approximation of $D := \alpha_D - \frac{a}{s}$

[Ou-Woerdeman-2019]

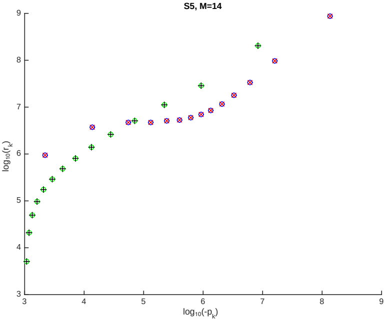
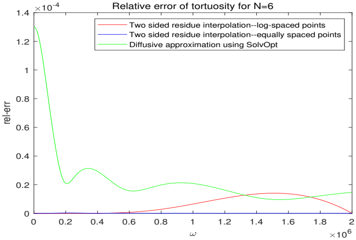


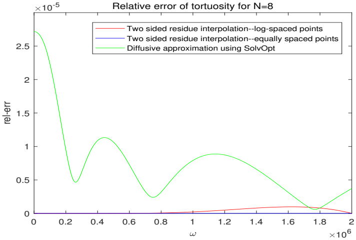
Figure: $(log_{10}(-p_k), log_{10}(r_k))$, $k = 1, .., 14$. Red x: Equally-spaced grid, Green circle: Log-spaced grids,$\omega \in [10^{-3}, 2 \times 10^6]$

Ou and Woerdeman, Operator Theory:Advances and Applications, Springer Nature, Vol. 272, pp. 341-362, 2019

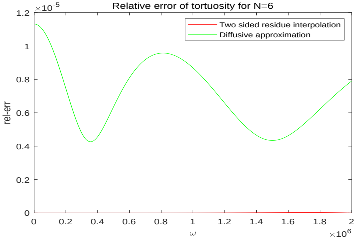
Comparision with state of the art [Xie-Ou-Xu-2019]



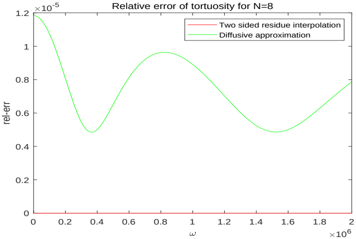
(a) N=6



(b) N=8



(c) J=6



(d) J=8

N			3	6	8	12
J3	augmented JKD	log-spaced points	9.976e-04	1.163e-05	5.790e-07	1.229e-09
		equally spaced points	1.814e-04	2.280e-08	1.851e-10	6.880e-16
	Biot DA	log-spaced points	3.587e-03	1.460e-05	8.354e-06	2.740e-07
J4	augmented JKD	log-spaced points	3.685e-05	8.057e-09	2.887e-11	4.627e-16
	Biot DA	log-spaced points	1.745e-03	5.936e-06	6.448e-06	1.010e-07

Table: Relative error of tortuosity approximation for materials J_3 and J_4 at the central frequency 200 kHz of a Wicker wavelet.

Jiangming Xie, MYO, Liwei Xu, A discontinuous Galerkin method for wave propagation in orthotropic poroelastic media with memory terms, Journal of Computational Physics 2019 (In press)

This works for non-JKD tortuosity, too, because of the results in Avellaneda-Torquato-91!

Motivation

Problem raised: How micro-structural information Γ play a role in determining the macroscopic property $\boldsymbol{K}^D$, the stationary permeability?
Our (Chuan Bi and I) strategy is to embed this problem into a larger system where there are two materials following the same physical law.

Literature

Luc Tartar (1980) derived the stationary Darcy's law by homogenization theory.
The permeability tensor $\boldsymbol{K}^D$ is defined as

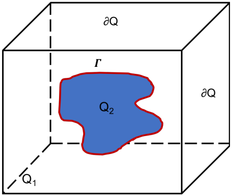
$$K_{i,j}^D = \frac{1}{|Q|} \int_{Q_1} (u_D^i)_j d\boldsymbol{y}$$

where $\boldsymbol{u}_D^i \in (H^1(Q_1))^3$ is the periodic solution to the cell problem

$$-\Delta \boldsymbol{u}_D^i + \nabla p_D = \boldsymbol{e}_i \quad \text{in } Q_1$$

$$\boldsymbol{u}_D^i \Big|_\Gamma = \boldsymbol{0}$$

$$\nabla \cdot \boldsymbol{u}_D^i = \boldsymbol{0} \quad \text{in } Q_1$$



Q_1 is the fluid domain, Γ the fluid-solid interface.

Strategy and existing works

- Lipton et al (1990) derived the self-permeability tensor $\boldsymbol{K}$ for the Darcy's law for viscous fluid flow passing stationary viscous bubbles. Two fluids with constant viscosities μ_1 and μ_2.
- Bruno et al. (1993) showed that the domain of analyticity for the deformation $\boldsymbol{u}(z)$ of a two-component elastic composite can be extended to $|z| \to \infty$ and $|z| \to 0$ where $z \in \mathbb{C}$ is the ratio between the material elastic properties.

To derived the IRF, we take the two steps
(1) Derive the IRF for the self-permeability K in Lipton's paper with two fluids $\mu_1 = 1$ and varying complex valued μ_2. It turned out the moments are determined by the case $\mu_1 = \mu_2$.
(2) Treat the permeability K^D as the limit case of $\mu_2 = \infty$. To do this, we prove that the support of measure is bounded away from ∞ by using the extension techniques in Bruno's paper to construct analytic solutions outside a large ball $|\mu_2| > C$.

Function spaces

Define the Hilbert space $H(Q)$ of admissible functions for the velocity

$$H(Q) = \left\{ \boldsymbol{v} : \boldsymbol{v} \in H^1(Q)^3 \middle| \operatorname{div}_{\boldsymbol{y}} \boldsymbol{v} = 0, \ \boldsymbol{v} \cdot \mathbf{n} = 0 \right. \\ \boldsymbol{v} \text{ is } Q\text{- periodic} \} \tag{9}$$

endowed with inner product

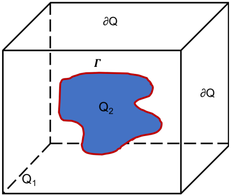
$$(\boldsymbol{u}, \boldsymbol{v})_Q = \int_Q 2\mu_1 e(\boldsymbol{u}) : \overline{e(\boldsymbol{v})} d\boldsymbol{y} \tag{10}$$

where $\mathbf{n}$ is the unit normal pointing inward towards Q_2, $e(\boldsymbol{u})$ is the strain tensor with $e(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla^T \boldsymbol{u})$.

The cell problem and existence of weak solution

The cell problem is to find solution $\boldsymbol{u}^k(\boldsymbol{y}; z) \in H(Q)$ such that

$$\begin{cases} \mathrm{div}_{\boldsymbol{y}} \left(2\tilde{\mu}(\boldsymbol{y}; z) e(\boldsymbol{u}^k) - p^k \boldsymbol{I}\right) + \boldsymbol{e}_k = \boldsymbol{0} \quad \text{in } Q_1 \cup Q_2 \\ [\![\boldsymbol{\pi}^k]\!]\mathbf{n} = \left([\![\boldsymbol{\pi}^k \mathbf{n}]\!] \cdot \mathbf{n}\right) \mathbf{n} \text{ on } \Gamma \end{cases}$$



where viscosity $\tilde{\mu}_{ijkl} = (\chi_1 \mu_1 + \chi_2 z \mu_1) I_{ijkl}$, fluid stress $\boldsymbol{\pi}^k = 2\tilde{\mu} e(\boldsymbol{u}^k) - p^k \boldsymbol{I}$.

- Existence and uniqueness of weak solution is ensured by Lax-Milgram lemma.
- Domain of analyticity is $z \in \mathbb{C} \backslash \{\Re z \leq 0\}$.

The self-permeability tensor $\boldsymbol{K}$

The self-permeability $\boldsymbol{K}$ (as opposed to Darcy's permeability $\boldsymbol{K}^D$) is defined as (Lipton '90)

$$K_{ij} = \frac{1}{|Q|} \int_Q u^i_j d\boldsymbol{y} \tag{11}$$

or equivalently, in terms of total energy:

$$K_{ij} = \frac{1}{|Q|} \int_Q 2\tilde{\mu}(\boldsymbol{y}; z) e(\boldsymbol{u}^i) : \overline{e(\boldsymbol{u}^j)} d\boldsymbol{y} \tag{12}$$

Extension of analyticity

We look for a form of Laurent series of $w = \frac{1}{z}$ near $w = 0$:

$$\boldsymbol{u}(\boldsymbol{y}; \boldsymbol{e}, w) = \sum_{k=0}^{\infty} \boldsymbol{u}_k(\boldsymbol{y}) w^k, \quad \boldsymbol{u}_k = \boldsymbol{u}_k^{in} + \boldsymbol{u}_k^{out} \tag{13}$$

where $\boldsymbol{u}_k^{in} \in H(Q_2)$, and $\boldsymbol{u}_k^{out} \in H(Q_1)$ denote the restrictions of velocity $\boldsymbol{u}_k$ inside and outside Γ. They satisfy the following PDEs for each order of $O(w^k)$:

	PDE	Interface condition
$\boldsymbol{u}_k^{in}$	$\sum_{k=0}^{\infty} w^k \left(\mathrm{div}_{\boldsymbol{y}} \left(\frac{2\mu_1}{w} e(\boldsymbol{u}_k^{in}) - p_k^{out} I\right)\right) = -\boldsymbol{e}$	$(\pi_{k-1}^{out} - \pi_k^{in})\mathbf{n} = \left((\pi_{k-1}^{out} - \pi_k^{in})\mathbf{n} \cdot \mathbf{n}\right) \mathbf{n}$
$\boldsymbol{u}_k^{out}$	$\sum_{k=0}^{\infty} w^k \left(\mathrm{div}_{\boldsymbol{y}} \left(2\mu_1 e(\boldsymbol{u}_k^{out}) - p_k^{out} I\right)\right) = -\boldsymbol{e}$	$\boldsymbol{u}_k^{out} = \boldsymbol{u}_k^{in}$

Two important lemmas #1: in→out

Lemma 1:
Let Q_2 be a connected, open bounded set of class $\mathcal{C}^2$ that does not intersect the boundary ∂Q. For any vector field $\boldsymbol{u}^{in} \in H(Q_2)$, there exists a unique weak solution $\boldsymbol{u}^{out}(\boldsymbol{y}; \boldsymbol{f}^{out}) \in H(Q_1)$ that satisfies the following Stokes equation with non-homogeneous boundary condition

$$\begin{cases} \text{div}_{\boldsymbol{y}} \left(2\mu_1 e(\boldsymbol{u}^{out}) - p^{out}\boldsymbol{I}\right) = \mathbf{f}^{out} & \text{in } Q_1 \\ \boldsymbol{u}^{out} = \boldsymbol{u}^{in} & \text{on } \Gamma \end{cases} \tag{14}$$

where in our context, $\mathbf{f}^{out} = \mathbf{0}$ or $\mathbf{f}^{out} = -\boldsymbol{e}_k$.
The solution $\mathbf{u}^{out}$ is bounded by

$$\left\|\boldsymbol{u}^{out}\right\|_{Q_1} \leq C_1 C_2 \left\|\mathbf{f}^{out}\right\|_{L^2(Q_1)} + 2C_0 \left\|\boldsymbol{u}^{in}\right\|_{Q_2}$$

where C_0, C_1, and C_2 depend only on the micro-structure.

Two important lemmas #2: out→in

Lemma 2:
Let Q_2 be a connected, open bounded set of class $\mathcal{C}^2$. For any pair of $(\boldsymbol{u}^{out}, p^{out}) \in H(Q_1) \times L^2(Q_1) \backslash \mathcal{R}$ that satisfies the PDE in **Lemma 1** with exerted force $\boldsymbol{f}^{out}$, there exists a unique vector field $\boldsymbol{u}^{in}(\boldsymbol{y}; \boldsymbol{f}^{in}) \in H(Q_2)$ that satisfies the Stokes equation with continuity of tangential traction on Γ

$$\begin{cases} \operatorname{div}_{\boldsymbol{y}} \left(2\mu_1 e(\boldsymbol{u}^{in}) - p^{in} \boldsymbol{I}\right) = \boldsymbol{f}^{in}, & \text{in } Q_2 \\ \left(\boldsymbol{\pi}^{out} - \boldsymbol{\pi}^{in}\right) \mathbf{n} = \left(\left(\boldsymbol{\pi}^{out} - \boldsymbol{\pi}^{in}\right) \mathbf{n} \cdot \mathbf{n}\right) \mathbf{n}, & \text{on } \Gamma \end{cases} \tag{15}$$

where in our context, $\mathbf{f}^{in} = \mathbf{0}$ or $\mathbf{f}^{in} = -\boldsymbol{e}_k$.
The solution $\mathbf{u}^{in}$ is bounded by

$$\left\| \boldsymbol{u}^{in} \right\|_{Q_2} \leq C_1 C_2 \left\| \boldsymbol{f}^{in} \right\|_{L^2(Q_2)} + C_0 C_1 C_2 \left\| \boldsymbol{f}^{out} \right\|_{L^2(Q_1)} + C_0 \left\| \boldsymbol{u}^{out} \right\|_{Q_1}$$

where C_0, C_1, and C_2 depend only on the micro-structure.

Extension procedure

1. $O(w^{-1})$: there exists a unique $\boldsymbol{u}_0^{in} \in H(Q_2)$ satisfying the PDE:

$$\begin{cases} \mathrm{div}_{\boldsymbol{y}} \left(2\mu_1 e(\boldsymbol{u}_0^{in})\right) = \mathbf{0} & \text{in } Q_2 \\ 2\mu_1 e(\boldsymbol{u}_0^{in})\mathbf{n} = C(\boldsymbol{y})\mathbf{n} & \text{on } \Gamma \end{cases} \tag{16}$$

2. $O(w^{0})$: there exists a unique $\boldsymbol{u}_0^{out} \in H(Q_1)$ satisfying the PDE:

$$\begin{cases} \mathrm{div}_{\boldsymbol{y}} \left(2\mu_1 e(\boldsymbol{u}_0^{out}) - p_0^{out}\boldsymbol{I}\right) = -\boldsymbol{e} & \text{in } Q_1 \\ \boldsymbol{u}_0^{out} = \boldsymbol{u}_0^{in} = \mathbf{0} & \text{on } \Gamma \end{cases} \tag{17}$$

3. $O(w^{0})$: there exists a unique $\boldsymbol{u}_1^{in} \in H(Q_2)$ satisfying the PDE:

$$\begin{cases} \mathrm{div}_{\boldsymbol{y}} \left(2\mu_1 e(\boldsymbol{u}_1^{in}) - p_0^{in}\boldsymbol{I}\right) = -\boldsymbol{e} & \text{in } Q_2 \\ \left(\boldsymbol{\pi}_0^{out} - \boldsymbol{\pi}_1^{in}\right)\mathbf{n} = \left(\left(\boldsymbol{\pi}_0^{out} - \boldsymbol{\pi}_1^{in}\right)\mathbf{n}\cdot\mathbf{n}\right)\mathbf{n} & \text{on } \Gamma \end{cases} \tag{18}$$

Extension procedure: induction step

For any $k \geq 1$:

1. Given $\boldsymbol{u}_k^{in} \in H(Q_2)$, there exists a unique $\boldsymbol{u}_k^{out}(\boldsymbol{y}) \in H(Q_1)$ that satisfies

$$\begin{cases} \mathrm{div}_{\boldsymbol{y}} \left(2\mu_1 e(\boldsymbol{u}_k^{out}) - p_k^{out} \boldsymbol{I}\right) = \boldsymbol{0} & \text{in } Q_1 \\ \boldsymbol{u}_k^{out} = \boldsymbol{u}_k^{in} & \text{on } \Gamma \end{cases} \tag{19}$$

2. Given $\boldsymbol{u}_k^{out} \in H(Q_1)$, there exists a unique $\boldsymbol{u}_{k+1}^{in}(\boldsymbol{y}) \in H(Q_2)$ that satisfies

$$\begin{cases} \mathrm{div}_{\boldsymbol{y}} \left(2\mu_1 e(\boldsymbol{u}_{k+1}^{in}) - p_k^{in} \boldsymbol{I}\right) = \boldsymbol{0} & \text{in } Q_2 \\ \left(\boldsymbol{\pi}_k^{out} - \boldsymbol{\pi}_{k+1}^{in}\right) \mathbf{n} = \left(\left(\boldsymbol{\pi}_k^{out} - \boldsymbol{\pi}_{k+1}^{in}\right) \mathbf{n} \cdot \mathbf{n}\right) \mathbf{n} & \text{on } \Gamma \end{cases} \tag{20}$$

Convergence Results

- On the complex disk $|w| \le \frac{R}{2C_0^2}$ with $R < 1$, the partial sums for $\boldsymbol{u}_k^{in}$ and $\boldsymbol{u}_k^{out}$

$$S_n^{in} = \sum_{k=0}^{n} \boldsymbol{u}_k^{in} w^k, \quad S_n^{out} = \sum_{k=0}^{n} \boldsymbol{u}_k^{out} w^k$$

  converge to unique analytic functions $\boldsymbol{u}_\infty^{in}(\boldsymbol{y}; \boldsymbol{e}, w) \in H(Q_2)$ and $\boldsymbol{u}_\infty^{out}(\boldsymbol{y}; \boldsymbol{e}, w) \in H(Q_1)$, as $n \to \infty$.
- $\boldsymbol{u}_\infty^{in} + \boldsymbol{u}_\infty^{out} \equiv \boldsymbol{u}(\boldsymbol{y}; \boldsymbol{e}, w)$ in the cell problem.
- As $w \to 0$
  1. $\boldsymbol{u}_\infty^{in}(\boldsymbol{y}; \boldsymbol{e}_i, w) \to \boldsymbol{0}$ uniformly in Q_2
  2. $\boldsymbol{u}_\infty^{out}(\boldsymbol{y}; \boldsymbol{e}_i, w) \to \boldsymbol{u}_D^i(\boldsymbol{y})$ uniformly in Q_1.
  3. The self-permeability $\boldsymbol{K}$ converges uniformly to the permeability tensor $\boldsymbol{K}^D$ in the classical derivation of Darcy's law at a rate of $\sqrt{|z|}$

Numerical Results for convergence *Bi-Ou-Zhang-2019*

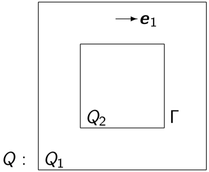


Table: Computed permeability K_{11}.

level	$\boldsymbol{K}^D$	$\boldsymbol{K}$		
		$z = 10^4$	$z = 1$	$z = 10^{-4}$
1	0.0105	0.0105	0.0122	0.0140
2	0.0119	0.0119	0.0144	0.0181
3	0.0125	0.0125	0.0154	0.0209
4	0.0128	0.0128	0.0159	0.0228
5	0.0129	0.0129	0.0161	0.0240

Representations formula

Define the self-adjoint operator for any $\boldsymbol{u} \in H(Q)$:

$$\Gamma_\chi \boldsymbol{u} = -\Delta_\#^{-1} \left(\mathrm{div}_{\boldsymbol{y}} \left(\chi_2 e(\boldsymbol{u})\right)\right)$$

$$K_{ij}(s) = \frac{s-1}{2\mu_1|Q|} \int_0^1 \int_Q \frac{\left(\tilde{M}(d\lambda)\Delta_\#^{-1}\boldsymbol{e}_i\right)_j}{s-1-\lambda} d\boldsymbol{y}$$

with measures

$$\tilde{\eta}_{ij}^{(\alpha)} = \frac{1}{2\mu_1|Q|} \int_Q \left(\left(\Gamma_\chi\right)^\alpha \Delta_\#^{-1}\boldsymbol{e}_i\right)_j d\boldsymbol{y} \tag{21}$$

$s = \frac{z}{z-1}$

Matching moments

Both series representation of $K_{i,j}$ should equal to each other, this yields

$$\eta_{ii}^{(\beta-1)} = -\tilde{\eta}_{ii}^{(\alpha)}, \quad \text{for } \alpha, \beta \geq 1$$

hence the dependence of micro-structure through Γ_χ become explicit, for example, $\alpha = \beta = 1$

$$\eta_{ii}^{0} = \left(-\frac{1}{|Q|} \int_Q 2\mu_1 e(\boldsymbol{u}^i(\boldsymbol{y};1)) : \overline{e(\boldsymbol{u}^i(\boldsymbol{y};1))} d\boldsymbol{y} \right) \left(\int_0^1 M_{ii}(d\lambda) \right)$$

$$\tilde{\eta}_{ii}^{1} = \frac{1}{|Q|} \int_Q 2\mu_1 \chi_2 e(\boldsymbol{u}^i(\boldsymbol{y};1)) : \overline{e(\boldsymbol{u}^i(\boldsymbol{y};1))} d\boldsymbol{y}$$

hence

$$\int_0^1 M_{ii}(d\lambda) = \frac{\int_Q 2\mu_1 \chi_2 e(\boldsymbol{u}^i(\boldsymbol{y};1)) : \overline{e(\boldsymbol{u}^i(\boldsymbol{y};1))} d\boldsymbol{y}}{\int_Q 2\mu_1 e(\boldsymbol{u}^i(\boldsymbol{y};1)) : \overline{e(\boldsymbol{u}^i(\boldsymbol{y};1))} d\boldsymbol{y}}$$