Reproducing kernel spaces and the Bargmann-Schiffer lemma

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Preliminaries

Nevanlinna functions appear in numerous settings, for instance dissipative continuous time linear systems, operator models,... (negative squares, several quaternionic variables, operator-valued, from a topological vector space to its dual, positive real lemma in the quaternionic setting)

Purpose of the talk is in a different avenue: it is a survey talk on how reproducing kernel methods allow to prove the Bargmann-Schiffer lemma. Allow to consider the non-positive case and the matrix-valued case.

Reproducing kernel approach to Schur algorithm and related problem, see A-Dym, *On applications of reproducing kernel spaces to the Schur algorithm and rational J unitary factorizations. Operator Theory: Advances and Applications*: OT18 (1986) pp. 89-159.

Sequence of papers with Dijksma and Langer on rks methods applied to generalized Schur and Nevanlinna functions. A very general setting (for the scalar case) was presented in:

D.A., A. Dijksma, H. Langer and D. Volok. *A Schur transformation for functions in a general class of domains*. Indagationes Mathematicae. (N.S) (2012), no. 4, pp. 921-969.
In principle one can get the present results from that paper.

Bargmann-Schiffer lemma

Let N be a non constant Nevanlinna, and $x_0 \in \mathbb{R}$ in a neighborhood of which N is smooth and $N'(x_0) > 0$. Then,

$$n(z) = -\frac{1}{N(z) - N(x_0)} + \frac{1}{N'(x_0)(z - z_0)}$$

is still a Nevanlinna function.

$$n(z) = -\frac{1}{N(z) - N(x_0)} - \left(-\frac{1}{N'(x_0)(z - z_0)}\right)$$

We thus get as a linear fractional transformation in terms of n. Leads to continuous fraction expansion of N

$$N(z) = N(x_0) + \frac{1}{\frac{1}{N'(x_0)(z - x_0)} - n}$$

Strategy

Build two reproducing kernel Hilbert spaces, one associated to N and one "to x_0" (i.e. "interpolation data") and relate them by an isometry which leads to a linear fractional transformation (rks approach, developped by A-Bruinsma-Dijksma-de Snoo, Dym)

Outline

Schur and Nevanlinna functions

Positive kernels

$K(z, w)$ defined in a set Ω is positive (also called positive definite) if all the matrices

$$(K(z_j, z_k))_{j,k=1}^N$$

are non negative (all choices of $N, z_1, \ldots, z_N \in \mathbb{N}$)

Reproducing kernel Hilbert space associated to a positive kernel:

$\mathcal{H}(K)$ with two properties: The functions $z \mapsto K(z, w) \in \mathcal{H}(K)$ and

$$f(w) = \langle f(\cdot), K(\cdot, w) \rangle_{\mathcal{H}(K)}$$

Negative squares

When the above matrices have at most k strictly negative eigenvalues. Then, the Hilbert space is replaced by a Pontryagin space.

A formula

Finite dimensional reproducing kernel spaces

Let $\mathcal{M}$ be a finite dimensional space of functions defined on a set Ω, with basis $f_1, \ldots, f_N$, and let P be an invertible positive matrix. Then $\mathcal{M}$ endowed wit the inner product

$$\langle F(z)c, F(z)d\rangle = d^*Pc, \quad F(z) = \begin{pmatrix} f_1(z) & \cdots & f_N(z) \end{pmatrix}, \quad c, d \in \mathbb{C}^N,$$

is a rkhs with reproducing kernel

$$F(z)P^{-1}F(w)^*$$

$N = 1$

$$K(z,w) = \frac{f_1(z)f_1(w)^*}{\|f\|^2}$$

Weak convergence in rks:

An important point in rkhs is that weak convergence implies pointwise convergence:

$$\lim_{n\to\infty} \langle f_n, g\rangle_{\mathcal{H}(K)} = \langle f, g\rangle_{\mathcal{H}(K)}$$

Take $g(z) = K(z, w)$ to get $\lim_{n\to\infty} f_n(w) = g(w)$.

Schur functions:

$|s(z)| \leq 1$ in $\mathbb{D}$ or equivalently $\frac{1-s(z)\overline{s(w)}}{1-z\overline{w}}$ is a positive kernel

Nevanlinna functions (also called *R* functions):

$\mathrm{Im} N(z) \geq 0$ in $\mathbb{C}_+$ or equivalently $\frac{N(z)-\overline{N(w)}}{z-\overline{w}}$ is a positive kernel in the open upper half-plane $\mathbb{C}_+$. An interesting link with physics in the paper of Wigner and von Neumann, *Significance of Loewner's theorem in the quantum theory of collisions*, Annals of math, 1954.

de Branges Rovnyak spaces

Reproducing kernel spaces associated to the kernels associated to a Schur function and a Nevanlinna function.

Schur

Schur algorithm:

If s is a Schur function and $|s(0)| < 1$ then

$$\sigma(z) = \begin{cases} \frac{s(z)-s(0)}{z(1-s(z)\overline{s(0)})}, & z \neq 0 \\ \frac{s'(0)}{1-|s(0)|^2}, & z = 0 \end{cases}$$

is also a Schur function. Can iterate and get a continued fraction expansion of s.

$$s(z) = \rho_0 + \cfrac{z(1-|\rho_0|^2)}{\overline{\rho_0}z - \cfrac{1}{\rho_1 + \cfrac{z(1-|\rho_1|^2)}{\overline{\rho_1}z - \ddots}}}.$$

Key in the proof is Schwarz' lemma.

Bargmann-Schiffer lemma

(called by Julia, generalized Schwarz lemma) Let N be Nevanlinna, and $x_0 \in \mathbb{R}$ in a neighborhood of which N is smooth and $N'(x_0) > 0$. Then,

$$n(z) = -\frac{1}{N(z) - N(x_0)} + \frac{1}{N'(x_0)(z - z_0)}$$

is still a Nevanlinna function.

Can be iterated and lead to continued fraction expansion

$\mathcal{L}(N)$ spaces

RKHS associated to a Nevanlinna function

$$N(z) = a + bz + \int_{\mathbb{R}} \left\{ \frac{1}{t-z} - \frac{t}{t^2+1} \right\} d\mu(t), \quad \int_{\mathbb{R}} \frac{d\mu(t)}{t^2+1} < \infty$$

and a real, $b \geq 0$. This formula gives an analytic extension of N to $\mathbb{C} \setminus \mathbb{R}$.

Moreover

$$\frac{N(z) - \overline{N(w)}}{z - \overline{w}} = b + \int_{\mathbb{R}} \frac{d\mu(t)}{(t-z)(t-\overline{w})} \quad \text{is positive definite in } \mathbb{C} \setminus \mathbb{R}$$

Remark:

Krein and Langer extended these representations to the non positive case.

The space $\mathcal{L}(N)$

$\mathcal{L}(N)$ is the space of functions analytic in $\mathbb{C} \setminus \mathbb{R}$ of the form

$$F(z) = \sqrt{b}c + \int_{\mathbb{R}} \frac{f(t)d\mu(t)}{t-z}$$

with norm

$$\|F\|^2 = b|c|^2 + \|f\|_\mu^2$$

Points of local losslessness

If $x_0 \in \mathbb{R}$ satisfies

$$\int_{\mathbb{R}} \frac{d\mu(t)}{(t-x_0)^{2m}} < \infty$$

(network terminology; Dewilde-Dym)

Lemma

Assume that $x_0 \in \mathbb{R}$ is such that $\int_{\mathbb{R}} \frac{d\mu(t)}{(t-x_0)^2} < \infty$. Then,

$$\lim_{h \to 0} N(x_0 + ih) \quad \text{exists}$$

the function $z \mapsto \frac{N(z)-N(x_0)}{z-x_0} \in \mathcal{L}(N)$ with norm $\sqrt{\int_{\mathbb{R}} \frac{d\mu(t)}{(t-x_0)^2}}$.

Idea of the proof (which works also for negative squares):

The functions

$$z \mapsto \frac{N(z) - N(x_0 + ih)}{z - (x_0 + ih)}$$

converge weakly and hence pointwise to $z \mapsto \frac{N(z)-N(x_0)}{z-x_0}$.

A one dimensional space

$$\mathcal{M} = \left\{ g(z) = \frac{\begin{pmatrix} 1 \\ N(x_0) \end{pmatrix}}{z - x_0} \right\}$$

The map $g \mapsto \begin{pmatrix} N(z) & -1 \end{pmatrix} g(z)$ sends $\mathcal{M}$ into $\mathcal{L}(N)$.

We endow $\mathcal{M}$ with the norm

$$\|g\|^2 = \int_{\mathbb{R}} \frac{d\mu(t)}{(t - x_0)^2}.$$

so that the map is an isometry.

$\mathcal{H}(\Theta)$ space

Let $J_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and

$$\Theta(z) = I - \frac{1}{p(z - x_0)} uu^* J_1, \quad p = \int_{\mathbb{R}} \frac{d\mu(t)}{(t - x_0)^2}, \quad u = \begin{pmatrix} 1 \\ N(x_0) \end{pmatrix}.$$

$$\frac{uu^*}{\|g\|^2 (z - x_0)(\overline{w} - x_0)} = \frac{J_1 - \Theta(z) J_1 \Theta(w)^*}{z - \overline{w}}$$

recognize the one dimensional reproducing kernel formula for $\mathcal{M}$.

We have an isometry from $\mathcal{M}$ into $\mathcal{L}(N)$. The space $\mathcal{M}$ has a special structure. It will force a linear fractional representation of N in terms of another Nevanlinna function.

Adjoint of multiplication operators in rkhs

Let X be a $\mathbb{C}^{q\times p}$-valued multiplier between the reproducing kernel Hilbert spaces $\mathcal{H}(K_1)$ and $\mathcal{H}(K_2)$ (K_1 is $\mathbb{C}^{p\times p}$-valued and K_2 is $\mathbb{C}^{q\times q}$-valued). Then,

$$(M_X^*(K_2(\cdot,w)d))(z) = K_1(z,w)X(w)^*d$$

So with $X(z) = \begin{pmatrix} N(z) & -1 \end{pmatrix}$

$$M_X^*\left(\frac{N(z)-\overline{N(w)}}{z-\overline{w}}\right) = \frac{J_1-\Theta(z)J_1\Theta(w)^*}{z-\overline{w}}\begin{pmatrix}\overline{N(w)}\\ -1\end{pmatrix}$$

The adjoint of an isometry is in particular contractive and so

$$\|M_X^*\left(\frac{N(z)-\overline{N(w)}}{z-\overline{w}}\right)\| \le \|\frac{J_1-\Theta(z)J_1\Theta(w)^*}{z-\overline{w}}\begin{pmatrix}\overline{N(w)}\\ -1\end{pmatrix}\|$$

$$\begin{pmatrix}N(z) & -1\end{pmatrix}\frac{J_1-\Theta(z)J_1\Theta(z)^*}{z-\overline{z}}\begin{pmatrix}\overline{N(z)}\\ -1\end{pmatrix} \le \frac{N(z)-\overline{N(z)}}{z-\overline{z}}$$

But

$$\begin{pmatrix}N(z) & -1\end{pmatrix}\frac{J_1}{z-\overline{z}}\begin{pmatrix}\overline{N(z)}\\ -1\end{pmatrix} = \begin{pmatrix}\overline{N(z)} & -1\end{pmatrix}\frac{\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}}{z-\overline{z}}\begin{pmatrix}\overline{N(z)}\\ -1\end{pmatrix}$$
$$= \frac{N(z)-\overline{N(z)}}{z-\overline{z}}$$

$$\begin{pmatrix} N(z) & -1 \end{pmatrix} \frac{\Theta(z) J_1 \Theta(z)^*}{z - \overline{z}} \begin{pmatrix} \overline{N(z)} \\ -1 \end{pmatrix} \geq 0.$$

Write (up to a multiplicative factor)

$$\Theta(z) = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix} = \begin{pmatrix} p(z - x_0) - N(x_0) & 1 \\ -N(x_0)^2 & p(z - x_0) + N(x_0) \end{pmatrix}$$

We have

$$\frac{\begin{pmatrix} Na - c & Nb - d \end{pmatrix} \begin{pmatrix} -\overline{Nb - d} \\ \overline{Na - c} \end{pmatrix}}{z - \overline{z}} \geq 0$$

and so

$$n = \frac{Nb - d}{Na - c}$$

is Nevanlinna

Bargmann-Schiffer

Plug in $n = \frac{Nb-d}{Na-c}$ the values of a, b, c, d to get

$$n(z) = \frac{N(z) - N(x_0) - p(z - x_0)}{-N(z)N(x_0) + N(z)p(z - x_0) + N(x_0)'}$$

and so

$$\frac{1}{\frac{1}{n(z)} + N(x_0)} = -\frac{1}{N(z) - N(x_0)} + \frac{1}{p(z - x_0)}$$

To get Bargmann-Schiffer itself, replace $\Theta(z)$ by $\Theta(z)\begin{pmatrix} 1 & 0 \\ N(x_0) & 1 \end{pmatrix}$.

Conclusions

The previous strategy can be applied to various generalizations of Nevanlinna functions. Tools from operator theory used still hold in the Pontryagin space setting. Also can consider matrix-valued case (tangential and full values). Similar strategy was used to study rigidity theorems for generalized Schur functions (A-Reich-Shoikhet, A-Dijksma-Langer-Reich-Shoikhet).