Conductivity imaging using Johnson-Nyquist noise

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Johnson's 1927 experiments: $|V(\omega)|^2$ proportional to $R(\omega)$

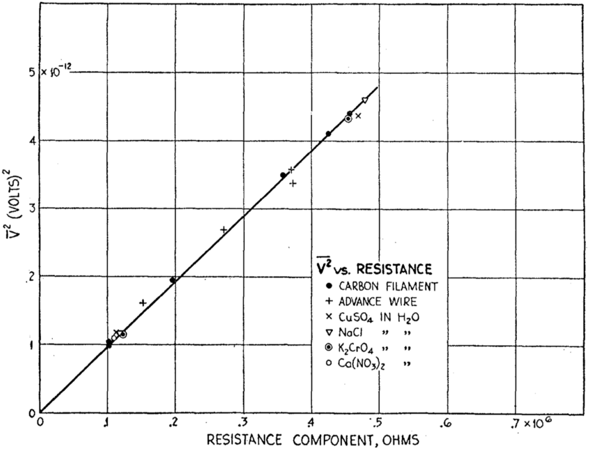


Fig. 4. Voltage-squared vs. resistance component for various kinds of conductors.

Source: Phys. Rev. **92**, 97 (1928)

Johnson's 1927 experiments: $|V(\omega)|^2/R(\omega)$ prop. to T

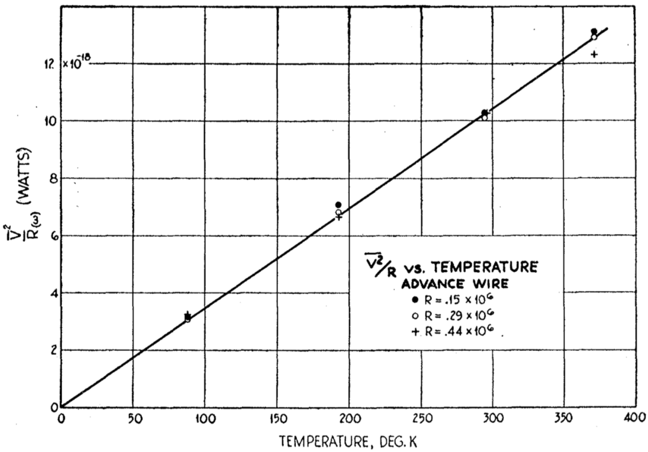
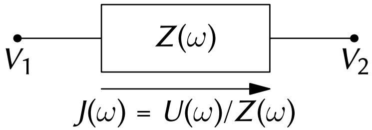


Fig. 6. Apparent power vs. temperature, for Advance wire resistances.

Source: Phys. Rev. **92**, 97 (1928)

Nyquist's 1928 explanation



- $J(\omega) =$ current
- $U(\omega) = V_2 - V_1 =$ voltage
- $Z(\omega) = R(\omega) + iX(\omega)$

At temperature T, thermally induced fluctuations of charges inside conductor give zero mean current and voltages:

$$\langle |J(\omega)|^2 \rangle = \frac{\kappa T}{\pi} \frac{R(\omega)}{|Z(\omega)|^2} \text{ and } \langle |U(\omega)|^2 \rangle = \frac{\kappa T}{\pi} R(\omega)$$

Here

- $\langle \cdot \rangle =$ statistical average
- $\omega = 2\pi =$ angular frequency
- $\kappa =$ Boltzmann constant $= 1.380649 \times 10^{-23} \, J \, K^{-1}$
- $\hbar =$ Planck's constant $= 6.62607015 \times 10^{-34} \, J \, s$
- Assumption: $\kappa T \gg \hbar\omega$

A matrix Langevin equation

Consider $\dfrac{d\boldsymbol{u}}{dt} = \boldsymbol{A}\boldsymbol{u} + \boldsymbol{f}(t)$, where

- $\boldsymbol{u}(t) \equiv$ state $\in \mathbb{C}^n$ at time t
- $\boldsymbol{A} \in \mathbb{C}^{n\times n}$, independent of time (for now)
- $\boldsymbol{f}(t) \equiv$ random "force" with:

$$\langle \boldsymbol{f}(t)\rangle = 0 \text{ and } \langle \boldsymbol{f}(t)\boldsymbol{f}(t')^*\rangle = 2\boldsymbol{B}\delta(t-t').$$

- The correlation matrix satisfies: $\boldsymbol{B} = \boldsymbol{B}^*$ and $\boldsymbol{B} \succeq 0$

If system reaches equilibrium:

- $\langle \boldsymbol{u}\rangle = 0$ and
- $\langle \boldsymbol{u}\boldsymbol{u}^*\rangle = \boldsymbol{M}$, with $\boldsymbol{M} = \boldsymbol{M}^*$ and $\boldsymbol{M} \succ 0$ (the strict inequality is assumed)

Fluctuation Dissipation Theorem:
$\langle \boldsymbol{f}\boldsymbol{f}^*\rangle$, $\boldsymbol{M}$ and "dissipative" part of $\boldsymbol{A}$ are related.

The Fluctuation Dissipation Theorem (FDT)

Split $\boldsymbol{A}$ into "symmetric" and "anti-symmetric" parts, i.e.

$$\boldsymbol{A} = \boldsymbol{A}_\mathrm{s} + \boldsymbol{A}_\mathrm{a} \text{ with } \boldsymbol{A}_\mathrm{s}\boldsymbol{M} = \boldsymbol{M}\boldsymbol{A}_\mathrm{s}^* \text{ and } \boldsymbol{A}_\mathrm{a}\boldsymbol{M} = -\boldsymbol{M}\boldsymbol{A}_\mathrm{a}^*$$

Theorem (Fluctuation dissipation theorem)

Correlation of the fluctuations must be equal to "dissipative" or symmetric part of $\boldsymbol{A}$:

$$\frac{1}{2}(\boldsymbol{A}\boldsymbol{M} + \boldsymbol{M}\boldsymbol{A}^*) = \boldsymbol{M}\boldsymbol{A}_s^* = \boldsymbol{A}_s\boldsymbol{M} = -\boldsymbol{B}.$$

(Callen and Welton 1951, Kubo 1966 and e.g. Zwanzig 2001)

Systems with memory

- In systems with memory:

$$\frac{d\boldsymbol{u}}{dt} = \boldsymbol{A}_a \boldsymbol{u} + \int_0^t \boldsymbol{A}_s(\tau)\boldsymbol{u}(t-\tau)d\tau + \boldsymbol{f}(t).$$

- Fluctuation Dissipation Theorem becomes

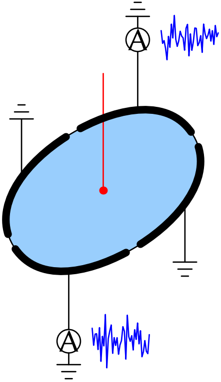
$$\langle \boldsymbol{f}(t)\boldsymbol{f}(t')^* \rangle = -\boldsymbol{A}_s(t-t')\boldsymbol{M},\ t \geq t'.$$

- In frequency:

$$\begin{aligned}\boldsymbol{A}_s[\omega]\boldsymbol{M} &= \int_0^\infty dt e^{-i\omega t}\boldsymbol{A}_s(t)\boldsymbol{M} \\ &= -\int_0^\infty dt e^{-i\omega t}\langle \boldsymbol{f}(0)\boldsymbol{f}(t)^* \rangle \\ &= \text{related to Herglotz-Nevanlinna function if system is real}\end{aligned}$$

- Can be applied to systems with losses (electric conduction, friction in particles moving in a fluid, elasticity, Maxwell equations,...)

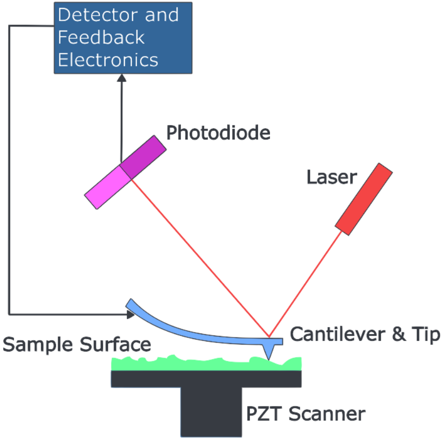
Setup



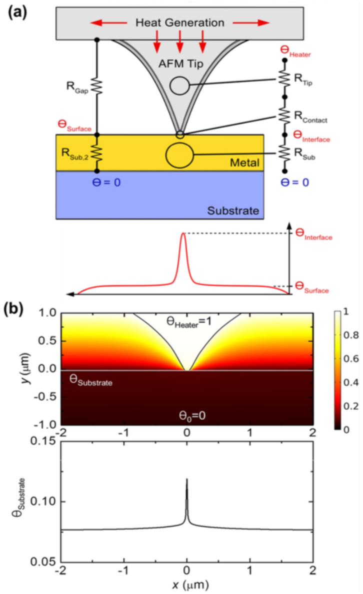
Main idea

- Heat a small portion of a conductive plate
- Measure thermal noise correlations at different locations of boundary
- Moving hotspot $\rightsquigarrow$ internal functional of σ

Possible application to Atomic Force Microscopy?



Source: Wikipedia



Source: King et al, ARHT, 2013

Fluctuational electrodynamics

In an isotropic, non-magnetic medium, thermal fluctuations can be modeled in the Maxwell equations by a random external electric current $\boldsymbol{j}_e$ with $\langle \boldsymbol{j}_e \rangle = 0$ (Rytov, Kravtsov, Tatarskii 1989)

$$\nabla \times \boldsymbol{H} = -ik\varepsilon \boldsymbol{E} + \frac{4\pi}{c}\boldsymbol{j}_e$$
$$\nabla \times \boldsymbol{E} = ik\mu \boldsymbol{H}$$

where

- electric permittivity is $\varepsilon(\boldsymbol{x},\omega) = \varepsilon'(\boldsymbol{x},\omega) + i\varepsilon''(\boldsymbol{x},\omega)$
- conductivity is $\sigma(\boldsymbol{x},\omega) = \omega\varepsilon''(\boldsymbol{x},\omega)/(4\pi)$
- magnetic permeability μ is assumed constant real
- wavenumber is $k = \omega/c$
- speed of light is c

Fluctuation Dissipation Theorem $\rightsquigarrow$ current fluctuations determined by conductivity, the dissipative part of ε

Random currents

- Thermally induced random currents are such that $\langle \boldsymbol{j}_e \rangle = 0$ and by FDT:

$$\left\langle \boldsymbol{j}_e(\boldsymbol{x},\omega)\boldsymbol{j}_e^*(\boldsymbol{x}',\omega)\right\rangle = -\frac{\Theta(\kappa,T)}{\pi}\mathrm{Re}\left(\frac{i\omega}{4\pi}\varepsilon(\boldsymbol{x},\omega)\right)\delta(\boldsymbol{x}-\boldsymbol{x}')\boldsymbol{I},$$

  where

$$\Theta(T,\omega) = \frac{\hbar\omega}{2}\coth\frac{\hbar\omega}{2\kappa T}$$

  is energy of a quantum oscillator.

- If $\kappa T \gg \omega$ we have $\Theta(T,\omega) \approx \kappa T$ (see e.g. Landau, Lifshitz 1960) and

$$\left\langle \boldsymbol{j}_e(\boldsymbol{x},\omega)\boldsymbol{j}_e^*(\boldsymbol{x}',\omega)\right\rangle = \frac{\kappa T}{\pi}\sigma(\boldsymbol{x},\omega)\delta(\boldsymbol{x}-\boldsymbol{x}')\boldsymbol{I}.$$

Quasistatic approximation: scalar model

When $\omega\mu|\varepsilon|L^2 \ll 1$, $L \equiv$ characteristic length, we get $\nabla \times \boldsymbol{E} \approx 0$. Taking $\boldsymbol{E} = -\nabla\phi$ (Cheney, Isaacson, Newell 1999)

$$\nabla \cdot [\widetilde{\sigma} \nabla \phi] = \nabla \cdot \boldsymbol{j}_e$$

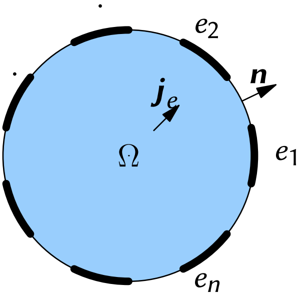
where

$$\widetilde{\sigma}(\boldsymbol{x},\omega) \equiv -\frac{i\omega}{4\pi}\varepsilon(\boldsymbol{x},\omega) = \sigma(\boldsymbol{x},\omega) - i\omega\frac{\varepsilon'(\boldsymbol{x},\omega)}{4\pi}$$

and

$$\langle \boldsymbol{j}_e(\boldsymbol{x},\omega)\boldsymbol{j}_e^*(\boldsymbol{x},\omega)\rangle = \frac{\kappa T}{\pi}\sigma(\boldsymbol{x},\omega)\delta(\boldsymbol{x}-\boldsymbol{x}')\boldsymbol{I}.$$

Simplified model



$$\begin{aligned} \nabla\cdot[\widetilde{\sigma}\nabla\phi] &= \nabla\cdot\boldsymbol{j}_e \text{ in } \Omega, \quad &(\boldsymbol{j}_e \text{ random from FDT})\\ \phi &= 0 \text{ on } \partial\Omega. \quad &(\text{grounding condition}). \end{aligned}$$

Measurements are $\langle \boldsymbol{J}\boldsymbol{J}^* \rangle$ where

$$\boldsymbol{J} = \left[\int_{\partial\Omega} e_1 \boldsymbol{j}\cdot\boldsymbol{n} dS, \ldots, \int_{\partial\Omega} e_N \boldsymbol{j}\cdot\boldsymbol{n} dS \right]^T.$$

Here $\boldsymbol{j} = \widetilde{\sigma}\nabla\phi$ and $e_1, \ldots, e_N$ are functions modelling "electrodes".

The covariance of the measurements

For $i = 1, \ldots, N$, consider the solutions to the auxiliary Dirichlet problems

$$\nabla \cdot [\widetilde{\sigma} \nabla u_i] = 0 \text{ in } \Omega,$$
$$u_i = e_i \text{ on } \partial\Omega.$$

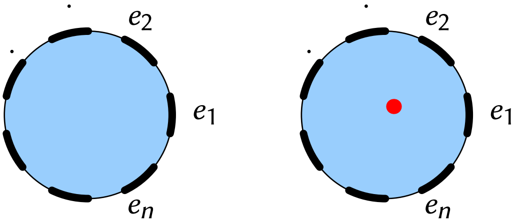
Theorem

$$[\langle \boldsymbol{J}\boldsymbol{J}^* \rangle]_{ij} = \frac{\kappa}{\pi} \int_\Omega d\boldsymbol{x} \sigma(\boldsymbol{x}) T(\boldsymbol{x}) \nabla u_i(\boldsymbol{x}) \cdot \nabla \overline{u_j}(\boldsymbol{x}).$$

- Proof using linearity of average $\langle \cdot \rangle$ and several integration by parts.
- Result holds with more realistic mixed Dirichlet and Neumann conditions to model insulation between electrodes
- Similar to Kirchhoff's law for far field heat transfer (Rytov, Kravtsov, Tatarskii 1988)

Getting an internal functional

With measurements:



$\left\langle \boldsymbol{J}_{T_0} \boldsymbol{J}^*_{T_0} \right\rangle$ $\qquad$ $\left\langle \boldsymbol{J}_{T_0+\delta T} \boldsymbol{J}^*_{T_0+\delta T} \right\rangle$

We can get

$$[\left\langle \boldsymbol{J}_{T_0+\delta T} \boldsymbol{J}^*_{T_0+\delta T} \right\rangle - \left\langle \boldsymbol{J}_{T_0} \boldsymbol{J}^*_{T_0} \right\rangle]_{ij} = \frac{\kappa}{\pi} \int_\Omega d\boldsymbol{x} \delta T(\boldsymbol{x}) \sigma(\boldsymbol{x}) \nabla u_i(\boldsymbol{x}) \cdot \nabla \overline{u_j}(\boldsymbol{x}).$$

By a sufficiently large basis of δT (beam position or other illumination patterns) we get the internal functional

$$H_{ij}(\boldsymbol{x}) = \sigma(\boldsymbol{x}) \nabla u_i(\boldsymbol{x}) \cdot \nabla \overline{u_j}(\boldsymbol{x}), \text{ for } \boldsymbol{x} \in \Omega.$$

Note: If $\widetilde{\sigma}$ is real, $H_{ii}(\boldsymbol{x}) =$ power dissipated at $\boldsymbol{x}$.

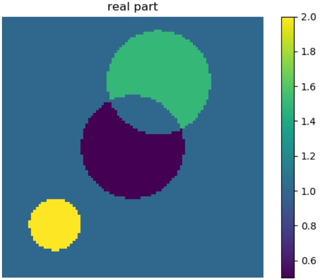
Stable reconstruction results and algorithms

The inverse problem of finding a **real** σ from $\sigma \nabla u_i(\boldsymbol{x}) \cdot \nabla u_j(\boldsymbol{x})$ appears in Ultrasound Modulated EIT or Acousto-Electric Tomography and is Lipschitz stable.

- Introduced: Ammari, Bonnetier, Capdebosq, Tanter and Fink 2008
- Lipschitz stability: Bal, Bonnetier, Monard, Triki 2011
- Linearization: Kuchment and Kunyanski, 2011
- General theory: Bal 2013
- Anisotropic σ: Bal, Guo, Monard, 2012-2014.

$\rightsquigarrow$ our problem is slightly different as the u_i depend on the $\text{Im}\,\widetilde{\sigma}$ and we can only perturb $\text{Re}\,\widetilde{\sigma}$.

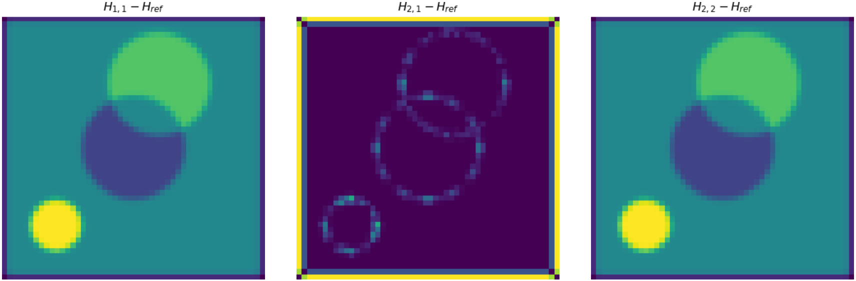
Numerical experiments (preliminary)



true (real part)

FD direct simulation with $e_1 = (x_1 + x_2)|_{\partial\Omega}$, $e_2 = (1 + x_1 - x_2)|_{\partial\Omega}$ on $\Omega = [0,1]^2$. Gaussian beam with std $= 10^{-4}$. $\varepsilon' = 1$, $\omega = 10KHz \times 2\pi$.

Internal functional data



FD direct simulation with $e_1 = (x_1 + x_2)|_{\partial\Omega}$, $e_2 = (1 + x_1 - x_2)|_{\partial\Omega}$ on $\Omega = [0,1]^2$. Gaussian beam with std = 10^{-4}. $\varepsilon' = 1$, $\omega = 10KHz \times 2\pi$.

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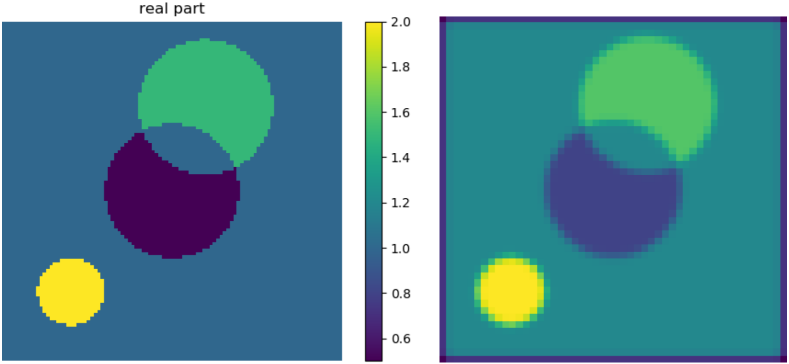

true $\qquad |H|$

FD direct simulation with $e_1 = (x_1 + x_2)|_{\partial\Omega}$, $e_2 = (1 + x_1 - x_2)|_{\partial\Omega}$ on $\Omega = [0,1]^2$. Gaussian beam with std $= 10^{-4}$. $\varepsilon' = 1$, $\omega = 10KHz \times 2\pi$.

Connection with Herglotz functions

Passivity $\implies$ permittivity at a fixed location $\boldsymbol{x}$ must obey (see e.g. Cassier, Milton 2017):

1. $\varepsilon(\boldsymbol{x},\omega)$ is analytic for $\operatorname{Im}(\omega) > 0$ + continuous when $\operatorname{Im}(\omega) = 0$
2. $\varepsilon(\boldsymbol{x},\omega) \to \varepsilon_\infty > 0$ when $|\omega| \to \infty$ and $\operatorname{Im}(\omega) \geq 0$
3. $\varepsilon(\boldsymbol{x},-\overline{\omega}) = \overline{\varepsilon(\boldsymbol{x},\omega)}$
4. $\operatorname{Im} \varepsilon(\boldsymbol{x},\omega) \geq 0$ when ω real and $\omega \geq 0$

The function $h(z) = z\varepsilon(\boldsymbol{x},\sqrt{z})$ is a Herglotz function (Milton, Eyre, Mantese 1997; Cassier, Milton 2017), i.e.

- h is analytic on $\mathbb{C}^+ = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ and
- $\operatorname{Im}(h(z)) \geq 0$ for $z \in \mathbb{C}^+$

Ideas

- If we can find $\epsilon(\boldsymbol{x},\omega)$ at some sampling frequencies $\omega_1,\ldots,\omega_n$ then we can use rational function approximation of $\varepsilon(\boldsymbol{x},z)$.
- Can we use Kramers-Kronigs relations or Herglotz function properties to "complete" data?

Summary and perspectives

- Thermal noise spatial correlations at one wavelength can be used to recover $\sigma \nabla u_i \cdot \nabla \overline{u_j}$
- When is the effect sufficiently large to be measured? $\rightsquigarrow$ Need more realistic numerical experiments with parameters from application.
- When $\widetilde{\sigma}$ is real, problem is equivalent to UMEIT $\rightsquigarrow$ many different reconstruction methods
- When $\widetilde{\sigma}$ is complex, no reconstruction method (yet). Perhaps rational function interpolation could help?
- We relied on equillibrium assumption, which may not hold because temperature gradients are large.