Asymptotics of skew standard Young tableaux

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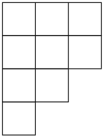
Outline

Basic definitions

- A partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ of n is a nonincreasing list of nonnegative integers of sum $|\lambda| = n$.
- It is identified with its Young diagram, formed by left-aligned row of boxes, with λ_1 boxes in the 1st row, λ_2 in the second, etc.



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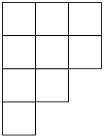
- It is identified with its Young diagram, formed by left-aligned row of boxes, with λ_1 boxes in the 1st row, λ_2 in the second, etc.
- A standard Young tableau (SYT) of shape λ is a filling of λ with integers from 1 to $|\lambda|$ with increasing rows and columns.
- Let f^λ denote the number of SYT of shape λ.



$\lambda = (3, 3, 2, 1)$

1	2	5
3	4	7
6	9	
8		

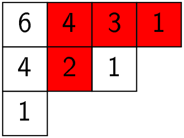
a SYT of shape λ

Hook length formula

Theorem (Hook length formula, Frame-Robinson-Thrall 1954)

For a straight shape λ,

$$f^{\lambda} = n! \prod_{\square \in \lambda} h(\square)^{-1}$$



hook lengths

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6	4	3	1
4	2	1	
1			

hook lengths

Asymptotics: Let λ be a diagram with at most $L\sqrt{n}$ rows and columns (called *balanced*). Most hook-lengths are of order $\Theta(\sqrt{n})$.

$$\log(f^\lambda) = \log(n!) - \tfrac{1}{2} n \log(n) - \sum_{\square \in \lambda} \log\left(\frac{h(\square)}{\sqrt{n}}\right)$$
$$= \tfrac{1}{2} n \log(n) + \mathcal{O}(n).$$

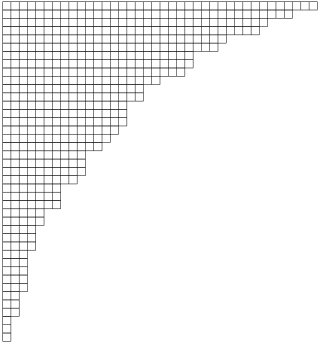
The $\mathcal{O}$ term can be written as an integral over the "limit shape" of λ.

Motivation from discrete probability theory

Plancherel measure on the set of Young diagrams of size n:

$$\mathbb{P}(\lambda) = \frac{(f^\lambda)^2}{n!}$$

(Vershik-Kerov, Logan-Shepp, 1977) The limit shape is the one that maximizes the $\mathcal{O}(n)$ term in the previous slide.



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Fix a *straight* shape λ and consider a uniform standard tableau T of shape λ (Romik-Pittel, Biane, Śniady, Sun, ...). Let $T^{(k)}$ be the diagram formed by boxes with entries at most k in T. Then

$$\mathbb{P}(T^{(k)} = \mu) = \frac{f^{\lambda/\mu} f^\mu}{f^\lambda}.$$

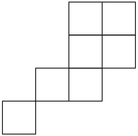
$\rightarrow$ we need the asymptotics of $f^{\lambda/\mu}$.

Outline

1. Standard Young tableaux
2. Skew standard Young tableaux
3. Proofs

Basic definitions

- The skew diagram λ/μ is obtained by removing the Young diagram of μ from the top-left corner of the Young diagram of λ.
  Notation: $n := |\lambda|$, $k := |\mu|$



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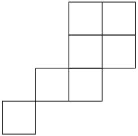

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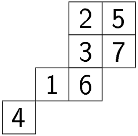

a skew SYT of shape λ/μ

Asymptotics for $|f^{\lambda/\mu}|$: previous results

- (Kerov, Stanley independently): asymptotic formula for μ fixed, $\frac{\lambda_i}{n} \to \alpha_i$, $\frac{\lambda'_i}{n} \to \beta_i$.

  In particular, when $\alpha_i = \beta_i = 0$ for all i (no rows or columns of size $\Theta(n)$), we have
  $$f^{\lambda/\mu} \sim \frac{f^\lambda f^\mu}{|\mu|!}.$$

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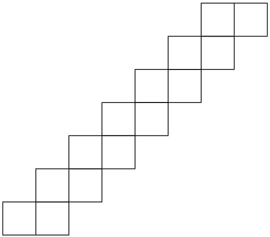
  Consequence: for a uniform random Young tableau T of shape λ:

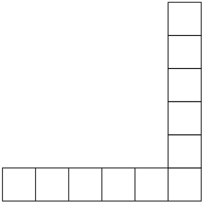
  $$\mathbb{P}(T^{(k)} = \mu) \sim \frac{(f^\mu)^2}{|\mu|!}.$$

  In other words, fixed size truncations are asymptotically Plancherel distributed.

Asymptotics for $|f^{\lambda/\mu}|$: previous results

- (Morales-Pak-Panova-Tassy): asymptotics for several families of shapes where $k, n-k = \Theta(n)$, all of the form
$$\log(f^{\lambda/\mu}) = \frac{1}{2}|\lambda/\mu| \log(|\lambda/\mu|) + \mathcal{O}(n),$$
with description of the $\mathcal{O}$ term.





Asymptotics for $|f^{\lambda/\mu}|$: our results

For simplicity, we assume λ and μ balanced. We set $A_{\lambda/\mu} := k!\frac{f^{\lambda/\mu}}{f^\lambda f^\mu}$.

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Theorem (D.–Féray 2017)

1. if $k = o(n^{1/3})$, then $A_{\lambda/\mu} = \sum_{\substack{\sigma \in S_k, \\ |\sigma| \le r}} \frac{\chi^{\lambda}(\sigma)}{f^{\lambda}} \frac{\chi^{\mu}(\sigma)}{f^{\mu}} + \mathcal{O}\left(\left(k^{\frac{3}{2}} n^{-\frac{1}{2}}\right)^{r+1}\right)$.
2. if $k = o(n^{1/2})$, then $A_{\lambda/\mu} \le \exp\left[\mathcal{O}\left(k^{3/2} n^{-1/2}\right)\right]$.
3. if $k \ge C n^{1/2}$, then $A_{\lambda/\mu} \le \exp\left[k \log \frac{k^2}{n} + \mathcal{O}(k)\right]$.

Here, r is a fixed integer and $|\sigma|$ denotes the absolute length of the permutation σ, i.e. the number of transpositions needed to factorize it.

Examples

If $k = o(n^{1/3})$, then $A_{\lambda/\mu} = \sum_{\substack{\sigma \in S_k,\\ |\sigma| \leq r}} \frac{\chi^\lambda(\sigma)}{f^\lambda} \frac{\chi^\mu(\sigma)}{f^\mu} + \mathcal{O}\left(\left(k^{\frac{3}{2}} n^{-\frac{1}{2}}\right)^{r+1}\right)$.

- For $r = 0$, the only permutation σ such that $|\sigma| \leq 0$ is Id, and $\chi^\lambda(Id) = f^\lambda$, so

$$A_{\lambda/\mu} = 1 + \mathcal{O}\left(k^{3/2} n^{-1/2}\right).$$

This generalises Kerov's and Stanley's results for fixed μ.

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- For $r = 1$, denote $b(\lambda) = \sum_{i \geq 1} (i-1)\lambda_i$, we have for a transposition τ

$$\frac{\chi^\lambda(\tau)}{f^\lambda} = \frac{2}{n(n-1)}(b(\lambda') - b(\lambda)).$$

Thus

$$A_{\lambda/\mu} = 1 + \frac{2}{n(n-1)}(b(\lambda') - b(\lambda))(b(\mu') - b(\mu)) + \mathcal{O}\left(k^3 n^{-1}\right).$$

How to get asymptotics for $f^{\lambda/\mu}$?

- No multiplicative formula in general;
  For some family of skew-shapes, $f^{\lambda/\mu}$ admits a product formula
  $\rightarrow$ convenient to see if a bound is sharp/make conjectures, but not to prove bounds...

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- Recent "additive" hook formula for skew shapes (Naruse), used in this context by Morales-Pak-Panova-Tassy.
- We will use representation theory instead (as Kerov-Stanley).

Outline

Branching rule and $f^{\lambda/\mu}$

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- Branching rule: restricting V_λ to $S_{n-1} \subseteq S_n$ we get:

$$\rho_\lambda / S_{n-1} \simeq \bigoplus_{\nu:\, \nu \nearrow \lambda} \rho_\nu .$$

$\nu \nearrow \lambda$ means $\nu \subseteq \lambda$ and $|\nu| = |\lambda| - 1$.

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Iterating the branching rule $r = n - k$ times gives:

$$\rho_\lambda / S_k \simeq \bigoplus_{\substack{\nu^{(0)},\dots,\nu^{(r-1)} \\ \nu^{(0)} \nearrow \dots \nearrow \lambda}} \rho_{\nu^{(0)}}$$

Sequences $\mu = \nu^{(0)} \nearrow \cdots \nearrow \nu^{(r)} = \lambda$ correspond to SYT of shape λ/μ.

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i.e. $f^{\lambda/\mu}$ is the multiplicity of ρ_μ in the restriction $\rho_\lambda \,/ S_k$.

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Corollary (Stanley 2001): $f^{\lambda/\mu} = \frac{1}{k!} \sum_{\sigma \in S_k} \chi^\lambda(\sigma) \chi^\mu(\sigma)$.
χ^λ: character (=trace) of the representation ρ_λ.

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$\rightarrow$ use asymptotic results for character values to get asymptotics for $f^{\lambda/\mu}$.

Bounds on symmetric group characters

Let $r(\nu)$, $c(\nu)$ denote the number of rows and columns of ν, respectively.

Theorem (Féray–Śniady, 2011)

There exists a constant $a > 1$, such that for every partition $\nu \vdash m$ and every permutation $\sigma \in S_m$,

$$\left|\frac{\chi^\nu(\sigma)}{f^\nu}\right| \leq \left[a \max\left(\frac{r(\nu)}{m}, \frac{c(\nu)}{m}, \frac{|\sigma|}{m}\right)\right]^{|\sigma|}.$$

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When ν is balanced (i.e. $r(\nu), c(\nu) \leq L\sqrt{m}$ for some L) , there are two regimes:

1. if $|\sigma| \leq L\sqrt{m}$, then $\frac{\chi^\nu(\sigma)}{f^\nu} \leq \left(\frac{aL}{\sqrt{m}}\right)^{|\sigma|}$;
2. if $|\sigma| > L\sqrt{m}$, then $\frac{\chi^\nu(\sigma)}{f^\nu} \leq \left(\frac{a|\sigma|}{m}\right)^{|\sigma|}$.

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- For fixed $|\sigma|$, the bound is optimal up to a multiplicative constant.
- For large $|\sigma|$, it's very bad: LHS is known to be at most 1, while the RHS grows exponentially in m.

Proof of the asymptotic expansion of $A_{\lambda/\mu}$ for $k = o(n^{1/3})$

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$$A_{\lambda/\mu} = k! \frac{f^{\lambda/\mu}}{f^\lambda f^\mu} = \sum_{\sigma \in S_k} \left(\frac{\chi^\lambda(\sigma)}{f^\lambda} \right) \left(\frac{\chi^\mu(\sigma)}{f^\mu} \right)$$

and want to apply the previous bound on characters.

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and want to apply the previous bound on characters.

- We have $|\sigma| \leq k = o(n^{1/3})$, so we always have $\left(\frac{\chi^{\lambda}(\sigma)}{f^{\lambda}}\right) \leq \left(\frac{aL}{\sqrt{n}}\right)^{|\sigma|}$;
- For $\left(\frac{\chi^{\mu}(\sigma)}{f^{\mu}}\right)$, it will depend on whether $|\sigma| \leq L\sqrt{k}$ or not.

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- For $\left(\frac{\chi^\mu(\sigma)}{f^\mu} \right)$, it will depend on whether $|\sigma| \leq L\sqrt{k}$ or not.

$$A_{\lambda/\mu} = \sum_{i=0}^{r} \sum_{\substack{\sigma \in S_k, \\ |\sigma| = i}} \frac{\chi^\lambda(\sigma)}{f^\lambda} \frac{\chi^\mu(\sigma)}{f^\mu} + S_1 + S_2,$$

where

$$S_1 = \sum_{i=r+1}^{L\sqrt{k}} \sum_{\substack{\sigma \in S_k, \\ |\sigma| = i}} \frac{\chi^\lambda(\sigma)}{f^\lambda} \frac{\chi^\mu(\sigma)}{f^\mu}, \qquad S_2 = \sum_{i=L\sqrt{k}+1}^{k} \sum_{\substack{\sigma \in S_k, \\ |\sigma| = i}} \frac{\chi^\lambda(\sigma)}{f^\lambda} \frac{\chi^\mu(\sigma)}{f^\mu}.$$

Proof of the asymptotic expansion of $A_{\lambda/\mu}$ for $k = o(n^{1/3})$

Lemma (Féray–Śniady 2011)

For all $k, i \in \mathbb{N}$, we have

$$\#\{\sigma \in S_k : |\sigma| = i\} \leq \frac{k^{2i}}{i!}.$$

Proof:
Every permutation in S_k appears exactly once in the product

$$[1 + (12)][1 + (13) + (23)] \cdots [1 + (1k) + \cdots + ((k-1)k)],$$

thus

$$\begin{aligned}\#\{\sigma \in S_k : |\sigma| = i\} &= [x^i](1+x)(1+2x)\cdots(1+(k-1)x) \\ &\leq [x^i](1+kx)^k = \binom{k}{i} k^i \leq \frac{k^{2i}}{i!}. \quad \square\end{aligned}$$

Proof of the asymptotic expansion of $A_{\lambda/\mu}$ for $k = o(n^{1/3})$

We can now bound S_1.

$$
\begin{aligned}
S_1 &= \sum_{i=r+1}^{L\sqrt{k}} \sum_{\substack{\sigma \in S_k, \\ |\sigma| = i}} \frac{\chi^\lambda(\sigma)}{f^\lambda} \frac{\chi^\mu(\sigma)}{f^\mu} \\
&\leq \sum_{i=r+1}^{L\sqrt{k}} \frac{k^{2i}}{i!} \left(\frac{aL}{\sqrt{n}} \right)^i \left(\frac{aL}{\sqrt{k}} \right)^i \\
&\leq \sum_{i=r+1}^{\infty} \frac{\left(a^2 L^2 k^{3/2} n^{-1/2}\right)^i}{i!} \\
&= \mathcal{O}\left(\left(k^{3/2} n^{-1/2} \right)^{r+1} \right),
\end{aligned}
$$

where the last bound is obtained as the tail of an exponential series.

Proof of the asymptotic expansion of $A_{\lambda/\mu}$ for $k = o(n^{1/3})$

We can now bound S_1.

$$S_1 = \sum_{i=r+1}^{L\sqrt{k}} \sum_{\substack{\sigma \in S_k, \\ |\sigma|=i}} \frac{\chi^\lambda(\sigma)}{f^\lambda} \frac{\chi^\mu(\sigma)}{f^\mu}$$

$$\leq \sum_{i=r+1}^{L\sqrt{k}} \frac{k^{2i}}{i!} \left(\frac{aL}{\sqrt{n}}\right)^i \left(\frac{aL}{\sqrt{k}}\right)^i$$

$$\leq \sum_{i=r+1}^{\infty} \frac{\left(a^2 L^2 k^{3/2} n^{-1/2}\right)^i}{i!}$$

$$= \mathcal{O}\left(\left(k^{3/2} n^{-1/2}\right)^{r+1}\right),$$

where the last bound is obtained as the tail of an exponential series. This is the error bound in our asymptotic expansion.

Proof of the asymptotic expansion of $A_{\lambda/\mu}$ for $k = o(n^{1/3})$

We can also bound S_2.

$$
\begin{aligned}
S_2 &= \sum_{i=L\sqrt{k}+1}^{k} \sum_{\substack{\sigma\in S_k,\\ |\sigma|=i}} \frac{\chi^\lambda(\sigma)}{f^\lambda}\frac{\chi^\mu(\sigma)}{f^\mu} \\
&\leq \sum_{i=L\sqrt{k}+1}^{k} \frac{k^{2i}}{i!}\left(\frac{aL}{\sqrt{n}}\right)^i\left(\frac{ai}{k}\right)^i \\
&\leq \sum_{i=L\sqrt{k}+1}^{k} \left(a^2 Lekn^{-1/2}\right)^i \qquad\qquad \text{by } i! \geq \frac{i^i}{e^i} \\
&\leq \left(a^2 Lekn^{-1/2}\right)^{L\sqrt{k}+1} \frac{1}{1-a^2 Lekn^{-1/2}}.
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where the last bound comes from the convergent geometric series.

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$$\leq \sum_{i=L\sqrt{k}+1}^{k} \frac{k^{2i}}{i!} \left(\frac{aL}{\sqrt{n}}\right)^i \left(\frac{ai}{k}\right)^i$$

$$\leq \sum_{i=L\sqrt{k}+1}^{k} \left(a^2 Lekn^{-1/2}\right)^i \qquad \text{by } i! \geq \frac{i^i}{e^i}$$

$$\leq \left(a^2 Lekn^{-1/2}\right)^{L\sqrt{k}+1} \frac{1}{1 - a^2 Lekn^{-1/2}}.$$

where the last bound comes from the convergent geometric series.
This is negligible compared to the bound for S_1. □

Proof that $A_{\lambda/\mu} \leq \exp\left[\mathcal{O}\left(k^{3/2}n^{-1/2}\right)\right]$ for $k = o(n^{1/2})$

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Recall that

$$A_{\lambda/\mu} = k! \frac{f^{\lambda/\mu}}{f^{\lambda} f^{\mu}} = \sum_{\sigma \in S_k} \left(\frac{\chi^{\lambda}(\sigma)}{f^{\lambda}}\right) \left(\frac{\chi^{\mu}(\sigma)}{f^{\mu}}\right).$$

We now write

$$A_{\lambda/\mu} = S_1' + S_2,$$

where

$$S_1' = \sum_{i=0}^{L\sqrt{k}} \sum_{\substack{\sigma \in S_k, \\ |\sigma| = i}} \frac{\chi^{\lambda}(\sigma)}{f^{\lambda}} \frac{\chi^{\mu}(\sigma)}{f^{\mu}},$$

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S_2 is the same as before, and therefore negligible in front of S_1. □

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S_3 gives the dominant term. □

Improving the bounds?

- We proved: when $k = o(n^{1/2})$,

$$A_{\lambda/\mu} \leq \exp\left[\mathcal{O}\left(k^{3/2}n^{-1/2}\right)\right].$$

  Moreover, we can find families of shapes λ/μ with $k = n^{\alpha}$, (for various $\alpha \in (0, 1/2)$) for which $\log(A_{\lambda/\mu})$ is of order $\Theta(k^{3/2}n^{-1/2})$.
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- When $k \geq Cn^{1/2}$, we proved $A_{\lambda/\mu} \leq \exp\left[k \log \frac{k^2}{n} + \mathcal{O}(k)\right]$.
Experimentally, $\log(A_{\lambda/\mu})$ is again at most of order $\Theta(k^{3/2}n^{-1/2})$.
$\rightarrow$ This bound is very likely not sharp.

Improving the bounds? Not with current bounds for characters

Assume $k \geq Cn^{1/2}$.

Call $U_{\mathsf{R}}(\sigma,\nu)$ (resp. $U_{\mathsf{MSP}}(\sigma,\nu)$, $U_{\mathsf{LS}}(\sigma,\nu)$ and $U_{\mathsf{F\acute{S}}}(\sigma,\nu)$) the upper bounds of Roichman (resp. Müller–Schlage-Putch, Larsen–Shalev, and Féray–Śniady) for $\left|\frac{\chi^\nu(\sigma)}{f^\nu}\right|$ and set

$$U_{\mathsf{best}}(\sigma,\nu) = \min\left(U_{\mathsf{R}}(\sigma,\nu), U_{\mathsf{MSP}}(\sigma,\nu), U_{\mathsf{LS}}(\sigma,\nu), U_{\mathsf{F\acute{S}}}(\sigma,\nu)\right),$$

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Proposition (D.–Féray, 2017)

$$\sum_{\sigma \in S_k} U_{best}(\sigma,\lambda) U_{best}(\sigma,\mu) \geq \exp\left[k \log \frac{k^2}{n} + \mathcal{O}(k)\right]$$

→ Even combining various bounds from the literature does not improve our result.

Improving the bounds?

Conjecture (D.–Féray, 2017)

There exists $C = C(L)$ such that for any balanced λ and μ, we have

$$\exp\left[-C\,k^{3/2}n^{-1/2}\right] \leq A_{\lambda/\mu} \leq \exp\left[C\,k^{3/2}n^{-1/2}\right],$$

- For $k = o(n^{1/3})$, this corresponds to our result;
- For $k = o(n^{1/2})$, we only have the upper bound;
- For $k \geq Cn^{1/2}$, we only have a weaker upper bound (and no lower bound).

Thank you for your attention!