Asymptotics of traces of paths in graded graphs.

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Graded graphs and measures on paths

- $V = V_0 \sqcup V_1 \sqcup \dots$
- $E = E_0 \sqcup E_1 \sqcup \dots$, $E_i = E(V_i, V_{i+1})$
- For $v \in V$, $\dim(v)$ is the number of paths from V_0 to v.
- **Path space** $\mathcal{P}(G)$: the space of infinite paths started at V_0
- Probabilistic measure μ on $\mathcal{P}(G)$ is *central*, if for each vertex $v \in G$ the probabilities to come to v using all possible paths from V_0 to v are mutually equal (to $1/\dim(v)$).

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Plancherel measure: Vershik's approach

- $n > m$, $v \in V_n$. Paths to v induce the measure ν_v^m on V_m.
- *irregularity function* $\mathrm{irreg}(v) : V \mapsto (0, 1]$
- the measures ν_v^m have a limit when $\mathrm{irreg}(v)$ tends to 0.
- Suppose that for each fixed m the measures ν_v^m have a limit when $\mathrm{irreg}(v) \to 0$. Denote by Pl_m the limit probability measure on V_m.

  Assume: the central measure μ on $\mathcal{P}(G)$ satisfies the **regularity condition**:

$$\liminf_n \mathbb{E}\,\mathrm{irreg}(v_n) = 0$$

  Then for every $m = 0, 1, \ldots$ the measure induced by μ at the level V_m, coincides with Pl_m.

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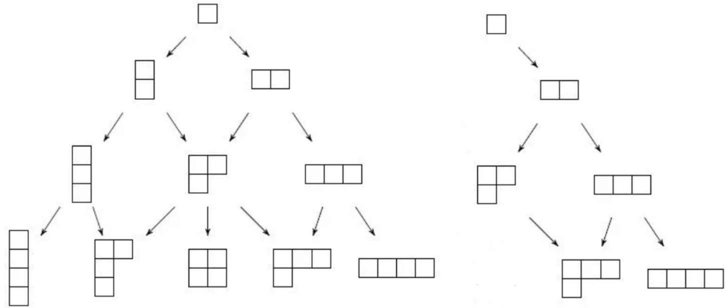

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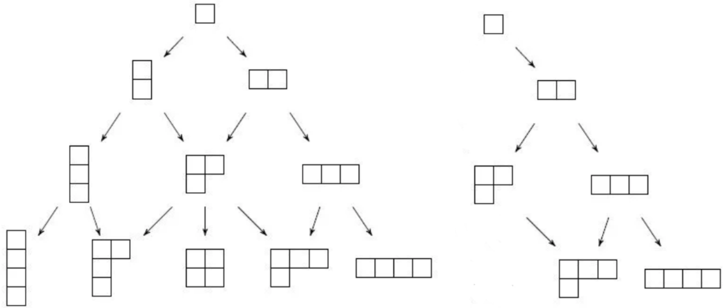

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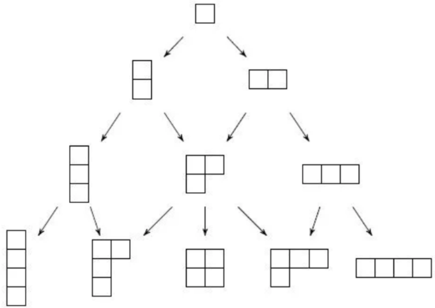

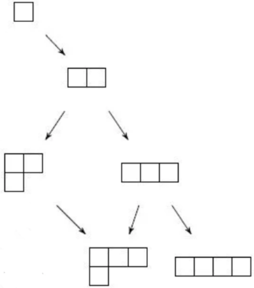

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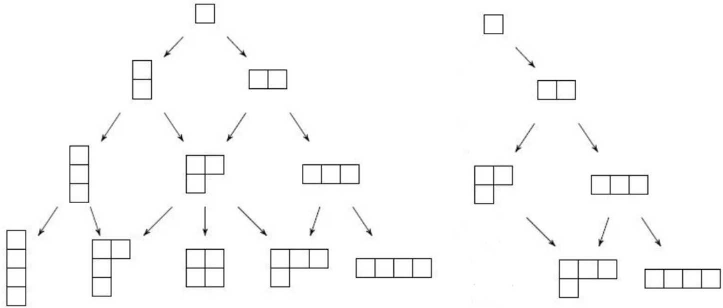

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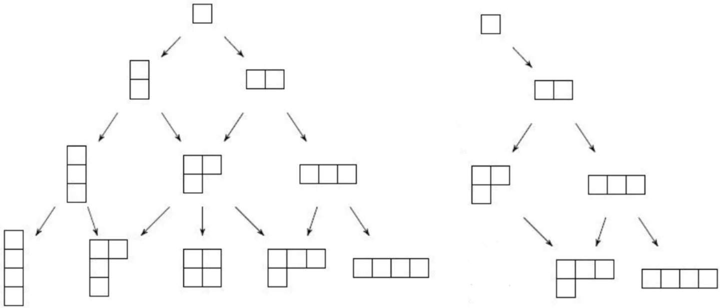

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- μ, λ — Young diagrams.
- $\dim(\lambda : \mu)$ the number of paths from μ to λ
- $0 \leqslant n_1 \leqslant n_2 - 1 \leqslant n_3 - 2 \leqslant \cdots \leqslant n_k - (k-1)$; the lengths of the rows of the diagram λ, m_i's for μ.
- $x^{\underline{n}} = x(x-1)\ldots(x-n+1)$ for arbitrary x and natural n.
- $a_\mu(x_1, \ldots, x_k) = \det(x_i^{m_j})_{1 \leqslant i,j \leqslant k}$;
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$$\dim(\lambda : \mu) = \frac{(|\lambda| - |\mu|)!}{\prod n_i!} b_\mu(n_1, \ldots, n_k).$$

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$$\dim(\lambda) = \prod_{i<j} \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j} \cdot \frac{n!}{\prod \lambda_i!} \quad \text{shifted hook length formula}$$

- two vertices λ, μ of the Schur graph, $\lambda_i \geqslant \mu_i$ for all i.
- $\ell = \ell(\mu)$ the number of rows of μ, that is $\mu_\ell > 0$, $\mu_{\ell+1} = 0$.
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$$\psi_\mu(x_1, \ldots, x_k) = \frac{1}{(k-\ell)!} \operatorname{Sym} \left(\prod_{i \le \ell} x_i^{\underline{\mu_i}} \prod_{i \le \ell, i<j} \frac{x_i + x_j}{x_i - x_j} \right)$$

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Comparing symmetric polynomials

- Let D be a positive integer and $x_1, x_2, \ldots, x_n$ be non-negative numbers such that each of them does not exceed $D^{-1} \sum x_i$.
- e_k is the sum of products of k-tuples of these numbers (elementary symmetric polynomial). Then

$$k! e_k \geqslant D^{\underline{k}} \cdot D^{-k} \cdot \left(\sum x_i\right)^k .$$

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Coefficients of symmetrizations

- $x_1, \ldots, x_n$; $V = V(x_1, \ldots, x_n) = \prod_{i<j}(x_j - x_i)$
- $\deg P(x_1, \ldots, x_n) = \binom{n}{2} = \deg V$

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$\deg F(x_1, \ldots, x_n) \leqslant m_1 + m_2 + \cdots + m_n$, where $m_i \geqslant 0$.
$C = [x_1^{m_1} \ldots x_n^{m_n}]F$.
Let A_1, A_2, ..., A_n be arbitrary subsets of the ground field F, $|A_i| = m_i + 1$ for all i.

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Example. $F = (x_1 + \cdots + x_n)^m \cdot V$, $\sum m_i = m + \binom{n}{2}$. Choose $A_i = \{0, 1, \ldots, m_i\}$, replace Σ^m to $(\Sigma - \binom{n}{2})^{\underline{m}}$. Only for $\alpha_i = m_i$ you get $F \neq 0$, thus

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Young graph $\dim(\lambda : \mu)$ asymptotics

$|\lambda| = n, |\mu| = m$, μ fixed, λ large and regular. Then

$$\dim(\lambda : \mu) \sim \frac{\dim(\lambda)\dim(\mu)}{m!}. \quad \dim(\lambda : \mu) = \frac{(n-m)!}{\prod n_i!} b_\mu(n_1, \dots, n_k).$$

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Young graph is "self-dual": the measure induced on V_m by paths from V_0 and from regular part of V_∞ coincide.

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$$\dim(\lambda : \mu) \sim 2^{m-\ell(\mu)} \frac{\dim(\lambda) \dim(\mu)}{m!}$$

$$\frac{\dim(\lambda : \mu)}{\dim(\lambda)} \sim n^{-m} \psi_\mu(\lambda_1, \lambda_2, \ldots, \lambda_k)$$

$$[x_1 x_2 \ldots x_m] \psi_\mu(x_1, \ldots) = 2^{m-\ell} \dim \mu.$$

$$\psi_\mu(x_1, \ldots, x_n) = \operatorname{Sym} F(x_1, \ldots, x_n) / V(x_1, \ldots, x_n), \quad \text{where}$$

$$F(x_1, \ldots, x_n) = \frac{1}{(n-\ell)!} \prod_{i \leqslant \ell} x_i^{\underline{\mu_i}} \prod_{i \leqslant \ell, i<j} (x_i + x_j) \prod_{\ell < i < j} (x_i - x_j),$$

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