Stochastic calculus techniques in the study of spread options: the optimal choice of strike conventions

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The objective, the model and notation

In this talk, we are interested in computing the exchange option price given by the expectation

$$V_0 = E(S_T^1 - S_T^2)_+.$$

by using a simple and (as far as possible) model-free technique.

The objective, the model and notation

Assume the following model for two asset prices S_1, S_2 (we take the interest rate $r = 0$ for the sake of simplicity)

$$\frac{dS_t^1}{S_t^1} = \lambda_1 \sigma_t dW_t^1$$
$$\frac{dS_t^2}{S_t^2} = \lambda_2 \sigma_t dW_t^2, \quad (1)$$

where W^1, W^2 are two Brownian motions and σ_t is a non-negative and square integrable process adapted to the filtration generated by another Brownian motion Z. We will use the notation

$$\left\langle W_t^1, Z \right\rangle = \rho_1, \left\langle W_t^2, Z \right\rangle = \rho_2, \left\langle W_t^1, W_t^2 \right\rangle = \rho.$$

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Margrabe's formula

It is well-known that, in the Black-Scholes setting, when $\sigma_t = \sigma$ is constant, the spread option price is given by

$$V_0 = BS\left(T, X_0^1, X_0^2, \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}\right),$$

where BS denotes the classical Black-Scholes function in terms of the log-prices $X_t^1 := \ln S_t^1$, $X_t^2 := \ln S_t^2$, $\sigma_1 = \lambda_1\sigma$, $\sigma_2 = \lambda_2\sigma$.

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A pricing approach

One approach to estimate spread option prices will consist in substituting σ_1 and σ_2 in the Margrabe formula by the implied volatilities of the corresponding vanilla options.

$$BS\left(T, x, y, \sqrt{I_1^2 + I_2^2 - 2\rho I_1 I_2}\right),$$

where $x := X_0^1$ and $y := X_0^2$.

But $I_1 = I(x, k_1)$ and $I_2 = I_2(y, k_2)$ depend on the strike ! ! ! ! and then the problem is

What is the optimal strike choice ?

Let us look again our problem

We want

$$V_0 = BS\left(T, x, y, \sqrt{I_1^2 + I_2^2 - 2\rho I_1 I_2}\right),$$

We denote

$$\hat{\gamma} := \sqrt{I_1^2 + I_2^2 - 2\rho I_1 I_2}$$

and we define $\gamma(x, y)$ as the spread implied volatility. That is, $\gamma(x, y)$ is the quantity such that

$$V_0 = BS\left(T, x, y, \gamma(x, y)\right).$$

Then, our problem reduces to find k_1, k_2 such that

$$\gamma(x, y) = \hat{\gamma}(x, y).$$

We will say that a pair $(k_1, k_2) \in L^2(\mathbb{R}^2; \mathbb{R}^2)$ is a short-time optimal strike convention of order n (a n-STOSC) if

$$\lim_{T \to 0} \frac{\partial^i \gamma}{\partial^i y}(x, x) = \lim_{T \to 0} \frac{\partial \hat{\gamma}^i}{\partial^i y}(x, x), \tag{2}$$

for any $i = 0, ..., n$.

Proposition

Assume $k_1(x,x) = k_2(x,x) = x$. *Then* (k_1, K_2) *is a* 0*-STOSC.*

Démonstration.

Results from Durrleman gives us that the ATM short time limit of the implied volatility is the spot volatility.

$$\lim_{T\to 0} \hat{\gamma}(x,x) = \tilde{\sigma}_0, \tag{3}$$

where

$$C(\lambda_1, \lambda_2, \rho) = \sqrt{\lambda_1^2 + \lambda_2^2 - 2\rho\lambda_1\lambda_2}$$

and

$$\tilde{\sigma}_0 = C(\lambda_1, \lambda_2, \rho)\,\sigma_0.$$

Similar arguments to those by Durrleman show that

$$\lim_{T\to 0} \gamma(x,x) = \tilde{\sigma}_0 \tag{4}$$

Theorem

Assume $k_1(x,x) = k_2(x,x) = x$ and that

$$\frac{\partial k_1}{\partial y}\frac{\partial l_1}{\partial z}(\lambda_1 - \rho\lambda_2) + \frac{\partial k_2}{\partial y}\frac{\partial l_2}{\partial z}(\lambda_2 - \rho\lambda_1) = \lambda_1\frac{\partial l_1}{\partial z} - \rho\lambda_1\frac{\partial l_2}{\partial z}. \quad (5)$$

Then (k_1, k_2) is a 1-STOSC.

Notice that this equation for $\frac{\partial k_1}{\partial y}$ and $\frac{\partial k_2}{\partial y}$ does not depend on the specific model for σ.

Proof

Similar as before, but instead of using previous results on the spot volatility, using previous results on the skew (see Alòs, León and Vives (2007) for vanillas and Alòs and León (2015) for spread options) we can see that

$$\frac{\partial \gamma(x,y)}{\partial y} \approx -\left(\frac{\lambda_1}{C(\lambda_1, \lambda_2, \rho)} \frac{\partial I_1}{\partial z}(x,x) - \frac{\lambda_2}{C(\lambda_1, \lambda_2, \rho)} \frac{\partial I_2}{\partial z}(x,x) \right). \tag{6}$$

and

$$\frac{\partial \hat{\gamma}(x,y)}{\partial y} \approx -\frac{1}{C(\lambda_1,\lambda_2,\rho)} \times \left[(\lambda_1 - \rho\lambda_2) \frac{\partial l_1}{\partial z} \frac{\partial k_1}{\partial y} + (\lambda_2 - \rho\lambda_1) \left(\frac{\partial l_2}{\partial z} \frac{\partial k_2}{\partial y} + \frac{\partial l_2}{\partial y} \right) \right], \tag{7}$$

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The case of linear log-strike conventions

Several strikes conventions have been proposed in the literature. Some classical examples (see for example Alexander and A. Venkatramanan (2011) and Swindle (2014)) are of the form

$$\begin{cases} k_1(x,y) = (1-a)x + ay \\ k_2(x,y) = ax + (1-a)y \end{cases}, \tag{8}$$

for some a, b are real numbers. For example, in Swindle (2014) the authors suggest to take $k_1 = \ln S_t^2$ and $k_2 = \ln S_t^1$. This choice corresponds to (8) in the case $a = 0$. On the other hand, in Alexander and A. Venkatramanan (2011) the authors studied the strike convention $k_1 = \ln S_t^1$ and $k_2 = \ln S_t^2$, that is the case $a = 1$

The case of linear log-strike conventions

Then $\frac{\partial k_1}{\partial y} = a$ and $\frac{\partial k_2}{\partial y} = 1 - a$ and (5) reads

$$a\left[\frac{\partial I_1}{\partial z}\left(1 - \frac{\rho\lambda_2}{\lambda_1}\right) - \frac{\partial I_2}{\partial z}\left(\frac{\lambda_2}{\lambda_1} - \rho\right)\right] = \frac{\partial I_1}{\partial z} - \frac{\lambda_2}{\lambda_1}\frac{\partial I_2}{\partial z}. \qquad (9)$$

Now, as $\frac{\partial I_1}{\partial z} = \frac{\rho_1}{\rho_2}\frac{\partial I_2}{\partial z}$ we get, after some algebra, that the optimal strike convention is given by

$$a = \frac{\lambda_1\rho_1 - \rho_2\lambda_2}{\rho_1(\lambda_1 - \rho\lambda_2) - \rho_2(\lambda_2 - \rho\lambda_1)} \qquad (10)$$

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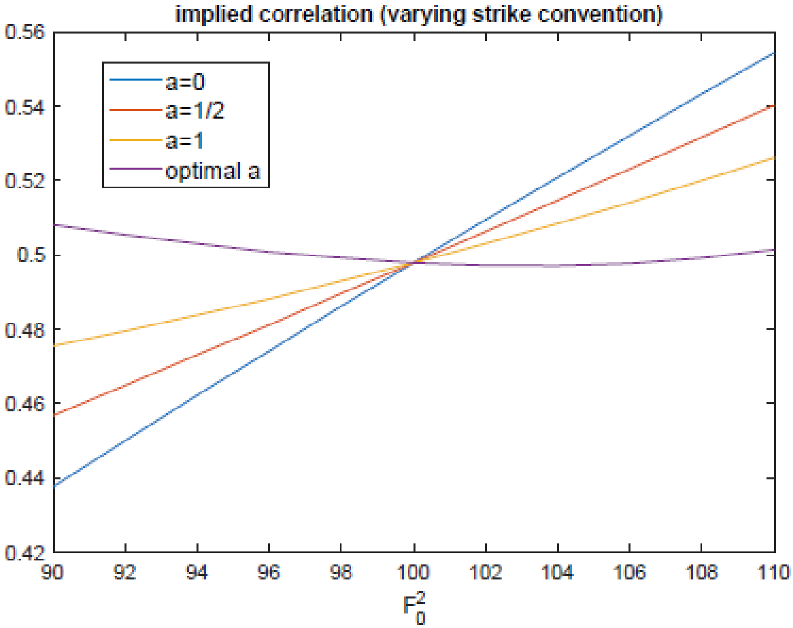
Numerical examples

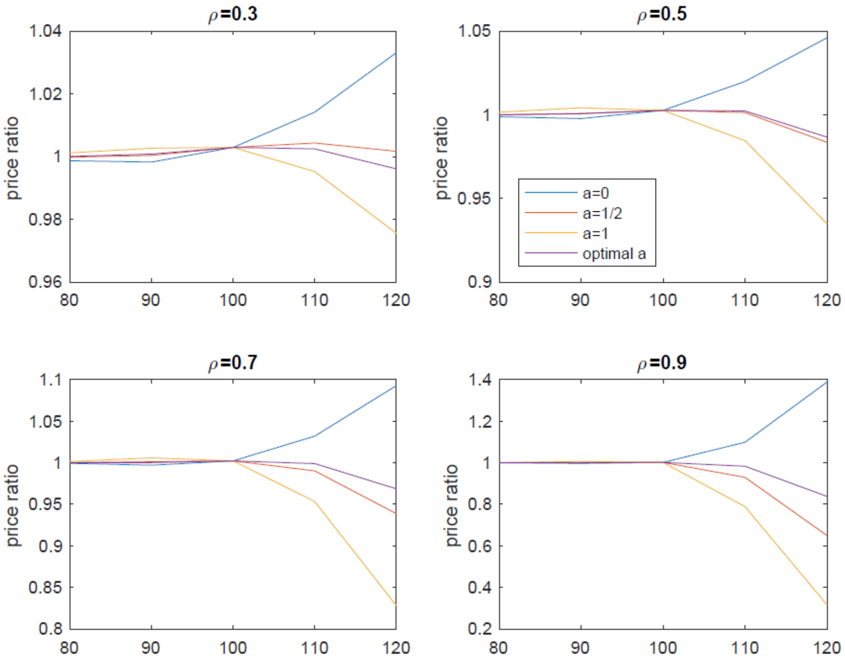
We set option maturity at $T = 0.05$ (a few weeks), and begin for simplicity with a Heston stochastic model given by

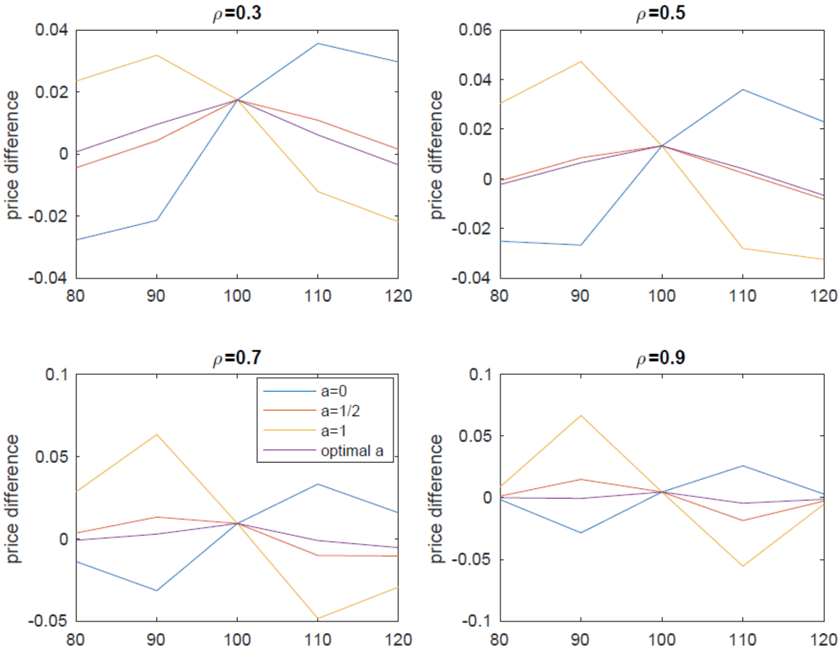
$$d\sigma_t^2 = \kappa \left(\theta - \sigma_t^2\right) dt + \nu \sqrt{\sigma_t^2} dW_t^{(3)},$$

along with the asset price model in (1), and the following set of parameters :

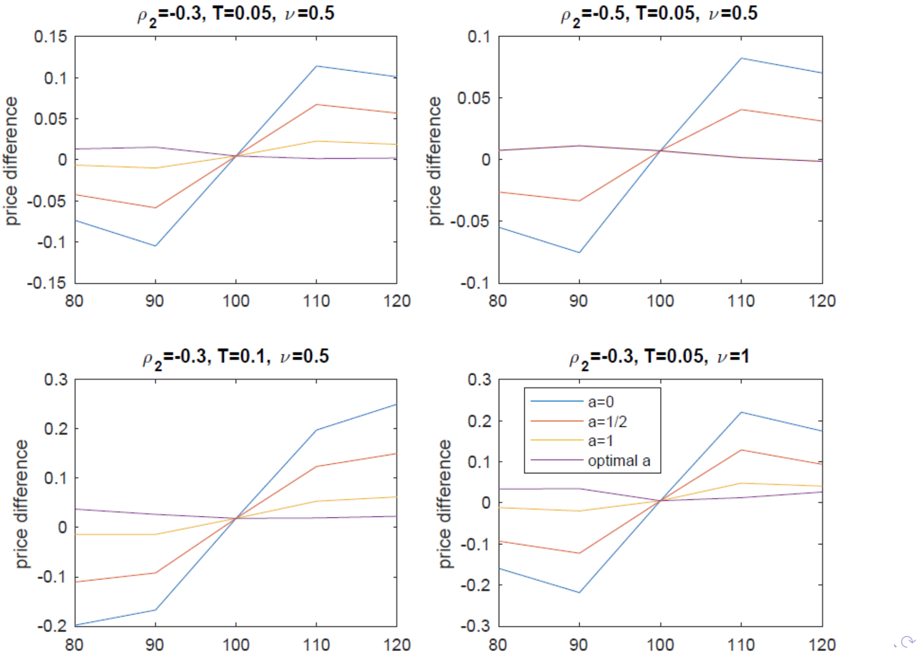
- volatility process (σ_t) parameters : $\kappa = 1.5, \theta = 0.15, \nu = 0.5, \sigma_0 = 0.15$
- volatility scaling factors : $\lambda_1 = 1.5, \lambda_2 = 1$
- correlation parameters $\rho = 0.5, \rho_{13} = -0.4, \rho_{23} = -0.5$

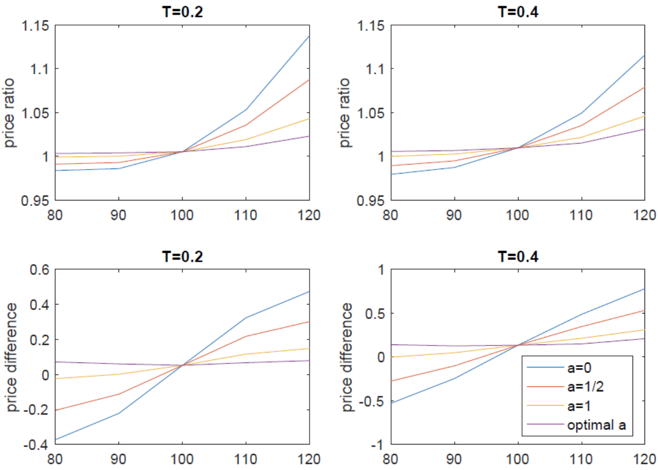

implied correlation (varying strike convention)
a=0
a=1/2
a=1
optimal a
0.56
0.54
0.52
0.5
0.48
0.46
0.44
0.42
90
92
94
96
98
100
102
104
106
108
110
F_0^2


ρ=0.3
ρ=0.5
ρ=0.7
ρ=0.9
price ratio
a=0
a=1/2
a=1
optimal a
1.04
1.02
1
0.98
0.96
1.05
0.95
0.9
1.1
0.85
0.8
1.4
1.2
0.8
0.6
0.4
0.2
80
90
100
110
120



ρ=0.3
ρ=0.5
ρ=0.7
ρ=0.9
price difference
a=0
a=1/2
a=1
optimal a
0.04
0.02
0
-0.02
-0.04
0.06
0.1
0.05
-0.05
-0.1
80
90
100
110
120


ρ_2=-0.3, T=0.05, ν=0.5
ρ_2=-0.5, T=0.05, ν=0.5
ρ_2=-0.3, T=0.1, ν=0.5
ρ_2=-0.3, T=0.05, ν=1
price difference
a=0
a=1/2
a=1
optimal a



Notice that it works for $T = 0.4$! ! !

Conclusions

We have presented a methodology to construct an optimal strike convention for spread options pricing in the context of stochastic volatility models.

This methodology is model-independent.

The obtained numerical results confirm the goodness of this technique.

Many thanks !