Exploiting Partial Correlations in Distributionally Robust Optimization

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Outline of Talk

1. Motivation: Distributionally Robust Appointment Scheduling

2. Moment Based Formulations

3. Exploiting Partial Correlations

4. Numerical Examples

Appointment Scheduling

- Random processing duration for patient $i \in [n]$ is $\tilde{u}_i$
- Scheduled duration for patient i is s_i where $s_0 = 0$
- Reporting time for patient i is $s_1 + s_2 + \ldots + s_{i-1}$
- Delay due to patient i is $\max(0, \tilde{u}_i - s_i)$
- Waiting time for patient i is $w_i = \max(w_{i-1} + \tilde{u}_{i-1} - s_{i-1}, 0)$

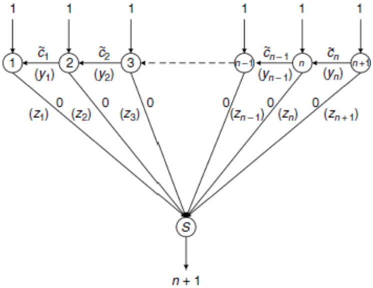
Appointment Scheduling

- Total waiting time of the patients and doctor's overtime

$$f(\tilde{\mathbf{u}}, \mathbf{s}) = \max(\tilde{u}_1 - s_1, 0) + \max(\tilde{u}_2 - s_2, \tilde{u}_2 - s_2 + \tilde{u}_1 - s_1, 0) + \dots$$
$$+ \max(\tilde{u}_n - s_n, \dots, \sum_{i=1}^{n} \tilde{u}_i - \sum_{i=1}^{n} s_i)$$

- Equivalent representation as the optimal objective of a network optimization problem with random arc lengths:

$$\begin{aligned}
\max\ & (\overbrace{\tilde{\mathbf{u}} - \mathbf{s}}^{\tilde{\mathbf{c}}})'\mathbf{y} \\
\text{s.t.}\ \ & y_i - y_{i-1} \geq -1,\ \ i = 2, \dots, n-1 \\
& y_n \leq 1, \\
& y_i \geq 0, \quad i = 1, \dots, n
\end{aligned}$$



Appointment Scheduling

- Seek a schedule to minimize the total expected waiting time and overtime (Gupta and Denton, 2008):

$$\min_{\mathbf{s} \in S} E_\theta[f(\tilde{\mathbf{u}}, \mathbf{s})]$$

- Challenges:
  - Specifying the joint probability distribution
  - Complexity of solving the resulting stochastic program
- Begen and Queyranne, 2011 - Integer valued, independent random processing durations:
  - Pseudo-polynomial time algorithm for computing the objective value for a fixed schedule (polynomial in the maximum processing duration)
  - Polynomial number of expected cost evaluations to find the optimal schedule using ideas from discrete convexity
- Generalizations to no-shows (Begen and Queyranne, 2011), sampling based approaches (Begen, Levi and Queyranne, 2012), piecewise linear cost functions (Ge, Wan, Wang and Zhang, 2014).

Distributionally Robust Appointment Scheduling

- Seek a schedule $\mathbf{s} \in S$ to minimize the worst-case sum of waiting times (Kong, Lee, Teo and Zheng, 2013):

$$\min_{\mathbf{s}\in S} \sup_{\theta\in\mathcal{P}} E[f(\tilde{\mathbf{u}}, \mathbf{s})]$$

- Set of feasible scheduled durations: $S = \{\mathbf{s} \ : \ s_i \geq 0, \sum_i s_i \leq T\}$.
- Summary of results:

$\mathcal{P}$	**Approach**	**Polynomial-time solvable**	**Tight**
Mean + Covariance (Kong, Lee, Teo and Zheng, 2013)	Copositive SDP relaxation	X ✓	✓ X
Mean + Variance (Mak, Rong and Zhang, 2015)	SOCP	✓	✓
Mean + Hypercube support + No-show (Bernoulli) (Jiang, Shen and Zhang, 2017)	LP	✓	✓
Mean + Bound on sum of variances and covariances (Bertsimas, Sim and Zhang, 2018)	SOCP	✓	X

Moments: Random Mixed Integer Linear Program

- Consider:

$$Z(\tilde{\mathbf{c}}) = \max\left\{\tilde{\mathbf{c}}'\mathbf{x} : \mathbf{x} \in \mathcal{X}\right\}$$

where $\mathcal{X}$ is the bounded feasible region to a MILP:

$$\mathcal{X} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq 0,\ x_j \in \mathcal{Z} \text{ for } j \in \mathcal{I} \subseteq [n]\}\,.$$

- Moment problem:

$$Z^*_{\text{full}}(\boldsymbol{\mu}, \boldsymbol{\Pi}) = \sup\left\{\mathbb{E}_\theta\left[Z(\tilde{\mathbf{c}})\right] :\ \mathbb{E}_\theta[\tilde{\mathbf{c}}] = \boldsymbol{\mu},\ \mathbb{E}_\theta[\tilde{\mathbf{c}}\tilde{\mathbf{c}}'] = \boldsymbol{\Pi},\ \theta \in \mathcal{P}(\mathbb{R}^n)\ \right\}.$$

- Other conic representable moment ambiguity sets - Delage and Ye (2010), Bertsimas, Doan, Natarajan, Teo (2010), Wiesemann, Kuhn and Sim (2014), ...

Moments: Completely Positive Program

- Given a closed convex cone $\mathcal{K}$, generalized completely positive cone over $\mathcal{K}$:

$$\mathcal{C}(\mathcal{K}) = \{\mathbf{A} \in \mathcal{S}^n : \exists \mathbf{b}_1, \ldots, \mathbf{b}_p \in \mathcal{K} \text{ such that } \mathbf{A} = \sum_{k \in [p]} \mathbf{b}_k \mathbf{b}_k'\}.$$

- Building on Burer (2010), Natarajan, Teo and Zheng (2011) provided an equivalent reformulation for 0-1 integer linear programs:

$$\begin{aligned}
Z^*_{\text{full}}(\boldsymbol{\mu}, \boldsymbol{\Pi}) \quad = \max_{\mathbf{p},\mathbf{X},\mathbf{Y}} \quad & \mathit{trace}(\mathbf{Y}) \\
\text{s.t} \quad & \begin{bmatrix} 1 & \boldsymbol{\mu}' & \mathbf{p}' \\ \boldsymbol{\mu} & \boldsymbol{\Pi} & \mathbf{Y}' \\ \mathbf{p} & \mathbf{Y} & \mathbf{X} \end{bmatrix} \in \mathcal{C}(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n_+), \\
& \mathbf{a}_k' \mathbf{p} = b_k, \qquad \forall k \in [p] \\
& \mathbf{a}_k' \mathbf{X} \mathbf{a}_k = b_k^2, \qquad \forall k \in [p] \\
& X_{jj} = x_j, \qquad \forall j \in \mathcal{I}.
\end{aligned}$$

Moments: Completely Positive Program

- General approach is to build on:

$$\mathbb{E}\left(\begin{bmatrix}1\\ \tilde{\mathbf{c}}\\ \mathbf{x}(\tilde{\mathbf{c}})\end{bmatrix}\begin{bmatrix}1\\ \tilde{\mathbf{c}}\\ \mathbf{x}(\tilde{\mathbf{c}})\end{bmatrix}'\right),$$

  where $\mathbf{x}(\tilde{\mathbf{c}})$ is a randomly chosen optimal solution for $\tilde{\mathbf{c}}$.
- Testing feasibility in the completely positive cone is NP-hard (Dickinson and Gibjen, 2014).
- Doubly nonnegative relaxation is often used for the completely positive cone - intersection of SDP and nonnegative cone
- Hanasusanto and Kuhn (2018), Xu and Burer (2018) provide copositive programs (dual formulation) for two-stage distributionally robust and robust linear programs with ambiguity set defined by a 2-Wasserstein ball around a discrete distribution and other uncertainty sets.

Moments: Large SDP

- Natarajan and Teo (2017) provide an alternate formulation based on convex hull of quadratic forms over the feasible region and SDP:

$$
\begin{aligned}
Z^*_{\text{full}}(\boldsymbol{\mu}, \boldsymbol{\Pi}) \quad = \max_{\mathbf{p},\mathbf{X},\mathbf{Y}} \quad & \mathit{trace}(\mathbf{Y}) \\
\text{s.t} \quad & \begin{bmatrix} 1 & \boldsymbol{\mu}' & \mathbf{p}' \\ \boldsymbol{\mu} & \boldsymbol{\Pi} & \mathbf{Y}' \\ \mathbf{p} & \mathbf{Y} & \mathbf{X} \end{bmatrix} \succeq 0, \\
& \left(\mathbf{p}, \mathbf{X}\right) \in \mathit{conv}\left\{\left(\mathbf{x}, \mathbf{x}\mathbf{x}'\right) : \mathbf{x} \in \mathcal{X}\right\}.
\end{aligned}
$$

- Characterizing the convex hull of quadratic forms is NP-hard for sets such as the Boolean quadric polytope with $\mathcal{X} = \{0, 1\}^n$ (Pitowsky, 1991)
- Identifying instances where this set is efficiently representable remains an active area of research (Anstreicher and Burer, 2010, Burer, 2015, Yang and Burer, 2018)

Exploiting Partial Correlations: Moments

- Information corresponding to non-overlapping moments
  - $\mathcal{N} = \{1, \ldots, n\}$
  - Non-overlapping subsets $\mathcal{N}_1, \ldots, \mathcal{N}_R$ of $\mathcal{N}$
  - Means μ^r, Second moments Π^r for $r = 1, \ldots, R$.
  - $n = 5, \mathcal{N}_1 = \{1, 2\}, \mathcal{N}_2 = \{3, 4, 5\}$
    $\mu^1 = [\mu_1, \mu_2]'$, $\mu^2 = [\mu_3, \mu_4, \mu_5]'$

$$\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} & \Pi_{13} & \Pi_{14} & \Pi_{15} \\ \Pi_{21} & \Pi_{22} & \Pi_{23} & \Pi_{24} & \Pi_{25} \\ \Pi_{31} & \Pi_{32} & \Pi_{33} & \Pi_{34} & \Pi_{35} \\ \Pi_{41} & \Pi_{42} & \Pi_{43} & \Pi_{44} & \Pi_{45} \\ \Pi_{51} & \Pi_{52} & \Pi_{53} & \Pi_{54} & \Pi_{55} \end{bmatrix} = \begin{bmatrix} \Pi^1 & ? \\ ? & \Pi^2 \end{bmatrix}$$

- Special case: Mean + Variance

$$\mathcal{N}_1 = \{1\}, \mathcal{N}_2 = \{2\}, \ldots, \mathcal{N}_n = \{n\}$$

- Special case: Mean + Covariance

$$\mathcal{N} = \{1, \ldots, n\}$$

Exploiting Partial Correlations: A Tight Formulation

Theorem

Define Z^* as the tight bound:

$$Z^* = \sup \left\{ \mathbb{E}_\theta \left[\max_{\mathbf{x}\in\mathcal{X}} \tilde{\mathbf{c}}'\mathbf{x} \right] : \ \mathbb{E}_\theta[\tilde{\mathbf{c}}] = \boldsymbol{\mu}, \ \mathbb{E}_\theta[\tilde{\mathbf{c}}^r(\tilde{\mathbf{c}}^r)'] = \mathbf{\Pi}^r \ \textit{for } r \in [R], \ \theta \in \mathcal{P}(\mathbb{R}^n) \right\}.$$

Define $\hat{Z}^*$ as the optimal objective value of the following semidefinite program:

$$\begin{aligned} \hat{Z}^* \quad = \max_{\mathbf{p},\mathbf{X}^r,\mathbf{Y}^r} \quad & \sum_{r=1}^{R} \textit{trace}(\mathbf{Y}^r) \\ \textit{s.t} \quad & \begin{bmatrix} 1 & \boldsymbol{\mu}^{r\prime} & \mathbf{p}^{r\prime} \\ \boldsymbol{\mu}^r & \mathbf{\Pi}^r & \mathbf{Y}^{r\prime} \\ \mathbf{p}^r & \mathbf{Y}^r & \mathbf{X}^r \end{bmatrix} \succeq 0, \qquad \textit{for } r \in [R], \\ & \left(\mathbf{p}, \mathbf{X}^1, \ldots, \mathbf{X}^R\right) \in \textit{conv}\left\{ \left(\mathbf{x}, \mathbf{x}^1\mathbf{x}^{1\prime}, \ldots, \mathbf{x}^R\mathbf{x}^{R\prime}\right) : \mathbf{x} \in \mathcal{X} \right\}. \end{aligned}$$

Then, $\hat{Z}^* = Z^*$.

Key Idea

- Using earlier result from Natarajan and Teo (2017):

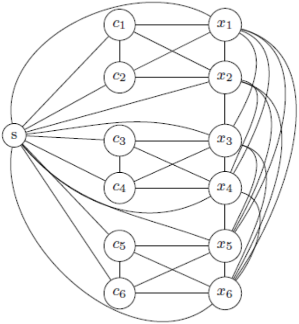
$$
\begin{aligned}
Z^* \quad = \max_{\mathbf{p},\mathbf{X},\mathbf{Y},\boldsymbol{\Delta}} \quad & trace(\mathbf{Y}) \\
\text{s.t} \quad & \begin{bmatrix} 1 & \boldsymbol{\mu}' & \mathbf{p}' \\ \boldsymbol{\mu} & \boldsymbol{\Delta} & \mathbf{Y}' \\ \mathbf{p} & \mathbf{Y} & \mathbf{X} \end{bmatrix} \succeq 0, \\
& \boldsymbol{\Delta}[\mathcal{N}_r] = \boldsymbol{\Pi}^r, \quad \text{for } r \in [R], \\
& \left(\mathbf{p}, \quad \mathbf{X}\right) \in conv\left\{\left(\mathbf{x}, \quad \mathbf{x}\mathbf{x}'\right) : \mathbf{x} \in \mathcal{X}\right\}.
\end{aligned}
$$

- $Z^* \leq \hat{Z}^*$ - straightforward
- $Z^* \geq \hat{Z}^*$ - exploit results from positive semidefinite matrix completion

Key Idea

- We need to complete the matrix given the optimal solution to $\hat{Z}^*$:

$$\mathbf{L}_p = \begin{bmatrix} 1 & \boldsymbol{\mu}^{1'} & \dots & \boldsymbol{\mu}^{R'} & \mathbf{p}_*^{1'} & \dots & \mathbf{p}_*^{R'} \\ \boldsymbol{\mu}^1 & \mathbf{\Pi}^1 & ? & ? & \mathbf{Y}_*^{1'} & ? & ? \\ \vdots & ? & \ddots & ? & ? & \ddots & ? \\ \boldsymbol{\mu}^R & ? & ? & \mathbf{\Pi}^R & ? & ? & \mathbf{Y}_*^{R'} \\ \mathbf{p}_*^1 & \mathbf{Y}_*^1 & ? & ? & & & \\ \vdots & ? & \ddots & ? & & \hat{\mathbf{X}} & \\ \mathbf{p}_*^R & ? & ? & \mathbf{Y}_*^R & & & \end{bmatrix}.$$



- Every partial positive semdefinite matrix with a pattern denoted by graph G has a positive semidefinite completion if and only if G is a chordal graph (Grone, Johnson, Sa and Wolkowicz, 1984).
- The matrix $\mathbf{L}_p$ has a positive semidefinite completion.

Special Case: Marginal Moments

- Assuming only knowledge of mean and variance:

$$
\begin{aligned}
Z^* \quad = \max_{p_i, X_{ii}, Y_{ii}} \quad & \sum_{i=1}^{n} Y_{ii} \\
\text{s.t} \quad & \begin{bmatrix} 1 & \mu_i & p_i \\ \mu_i & \Pi_{ii} & Y_{ii} \\ p_i & Y_{ii} & X_{ii} \end{bmatrix} \succeq 0, \qquad \text{for } i \in [n], \\
& \left(\mathbf{p}, X_{11}, \ldots, X_{nn}\right) \in conv \left\{ \left(\mathbf{x}, x_1^2, \ldots, x_n^2\right) : \mathbf{x} \in \mathcal{X} \right\}.
\end{aligned}
$$

- Characterizing this convex hull is hard for general polytopes; related to two-norm maximization over polytope (Freund and Orlin, 1985, Mangasarian and Shiau, 1986).
- For 0-1 polytopes with a compact representation, the bound is efficiently computable (Bertsimas, Natarajan and Teo, 2004).
- Mak, Rong and Zhang (2015) show that for the appointment scheduling problem, the bound is efficiently computable using an extended formulation for the network flow structure.

Appointment Scheduling (Adjoining Pairs of Patients)

- Computing the worst-case when correlations among service time durations of adjoining patients are known:

$$Z^*_{app}(\mathbf{s}) = \sup \left\{ \mathbb{E}_\theta \left[f(\tilde{\mathbf{u}}, \mathbf{s}) \right] : \mathbb{E}_\theta \left[\tilde{u}_i \right] = \mu_i, \mathbb{E}_\theta [\tilde{u}_i^2] = \Pi_{ii}, \text{ for } i \in [n], \right.$$
$$\left. \mathbb{E}_\theta [\tilde{u}_j \tilde{u}_{j+1}] = \Pi_{j,j+1}, \text{ for } j \in \{1, 3, ..., n-1\} \right\}.$$

- In the reduced formulation, we need to characterise $conv \left\{ \left[1, x_1, \ldots, x_n, x_1^2, \ldots, x_n^2, x_1 x_2, x_3 x_4, \ldots, x_{n-1} x_n\right] : \mathbf{x} \in \mathcal{X}_{app} \right\}$

Term	Mean+Variance	$\lvert\mathcal{N}_r\rvert = 2$	Mean+Covariance
x_i	✓	✓	✓
x_i^2	✓	✓	✓
$x_i x_{i+1}$		✓	✓
$x_i x_j$			✓

Appointment Scheduling (Adjoining Pairs of Patients)

Theorem

Given a schedule $\mathbf{s} \in S$*, we calculate the worst-case expected cost as follows:*

$$
\begin{aligned}
Z^*_{app}(\mathbf{s}) = \max_{p_i, X_{ij}, Y_{ij}, t_{kj}} \quad & \sum_{i=1}^{n} (Y_{ii} - s_i p_i) \\
\text{s.t.} \quad & \begin{bmatrix} 1 & \mu_i & \mu_{i+1} & p_i & p_{i+1} \\ \mu_i & \Pi_{ii} & \Pi_{i,i+1} & Y_{ii} & Y_{i,i+1} \\ \mu_{i+1} & \Pi_{i,i+1} & \Pi_{i+1,i+1} & Y_{i+1,i} & Y_{i+1,i+1} \\ p_i & Y_{ii} & Y_{i+1,i} & X_{ii} & X_{i,i+1} \\ p_{i+1} & Y_{i,i+1} & Y_{i+1,i+1} & X_{i,i+1} & X_{i+1,i+1} \end{bmatrix} \succeq 0, \quad \text{for } i \text{ odd}, \ i \in [n], \\
& p_i = \sum_{k=1}^{i} \sum_{j=i}^{n+1} t_{kj}(j - i), \quad \text{for } i \in [n], \\
& X_{ii} = \sum_{k=1}^{i} \sum_{j=i}^{n+1} t_{kj}(j - i)^2, \quad \text{for } i \in [n], \\
& X_{i,i+1} = X_{i+1,i} = \sum_{k=1}^{i} \sum_{j=i+1}^{n+1} t_{kj}(j - i)(j - (i+1)), \ \text{for } i \text{ odd}, \ i \in [n], \\
& \sum_{k=1}^{i} \sum_{j=i}^{n+1} t_{kj} = 1, \quad \text{for } i \in [n], \\
& t_{kj} \geq 0, \quad \text{for } 1 \leq k \leq j \leq n+1.
\end{aligned}
$$

Key Idea

- Polytope:

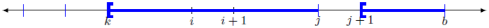
$$\left\{\mathbf{x} \in \mathbb{R}_+^n : x_i - x_{i-1} \geq -1,\ \ i = 2, ..., n-1, x_n \leq 1, x_i \geq 0, i \in [n]\right\}$$

- At every extreme point, either $x_i = 0$ or $x_i = x_{i+1} + 1$.
- Partition of intervals of integers in $\{1, 2, \ldots, n+1\}$ (Zangwill, 1966, 1969).
- Extreme points of the feasible region are given by:

$$\Bigg\{\mathbf{x} \in \mathbb{R}_+^n : x_i = \sum_{k=1}^{i} \sum_{j=i}^{n+1} T_{kj}(j-i), i \in [n], \sum_{k=1}^{i} \sum_{j=i}^{n+1} T_{kj} = 1, i \in [n],$$
$$T_{kj} \in \{0, 1\},\ \text{for } 1 \leq k \leq j \leq n+1\Bigg\}.$$

Key Idea

- Cross-terms: $x_i x_{i+1} = \sum_{k=1}^{i} \sum_{j=i+1}^{n+1} T_{kj}(j-i)(j-(i+1))$



- Convex hull of the set (exploit total unimodularity):

$$C_{app} = conv\Big\{ (p_1, \ldots, p_n, X_{11}, \ldots, X_{nn}, X_{12}, X_{34}, \ldots, X_{n-1,n}) \in \mathbb{R}^{5n/2} :$$

$$p_i = \sum_{k=1}^{i} \sum_{j=i}^{n+1} T_{kj}(j-i), \; X_{ii} = \sum_{k=1}^{i} \sum_{j=i}^{n+1} T_{kj}(j-i)^2, \quad \text{for } i \in [n],$$

$$X_{i,i+1} = \sum_{k=1}^{i} \sum_{j=i+1}^{n+1} T_{kj}(j-i)(j-(i+1)), \quad \text{for } i \in [n], i \text{ odd},$$

$$\sum_{k=1}^{i} \sum_{j=i}^{n+1} T_{kj} = 1, \; \text{for } i \in [n], \quad T_{kj} \in \{0,1\} \text{ for } 1 \le k \le j \le n+1 \Big\}.$$

Other Generalizations

- Project Evaluation and Review Technique (PERT) Networks: Maximum expected length of longest path in a graph under knowledge of partial moments
- Linear Assignment: Maximum expected total profit under knowledge of partial moments

Numerical Examples: Distributionally Robust Appointment Scheduling

Approaches

- Mean-variance
  - tight bound
  - polynomial size SOCP
- Doubly nonnegative relaxation
  - weaker upper bound
  - polynomial sized SDP relaxation
- Large-SDP
  - tight bound
  - not a polynomial sized SDP

- n random variables
- $\mu_i \sim \mathcal{U}[-2, 2] \ \forall i \in [n]$
- $\sigma_i \sim \mathcal{U}(0, 5] \ \forall i \in [n]$
- Correlation matrix:

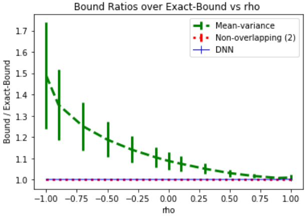
$$\begin{bmatrix} 1 & \rho & ? & ? & \ldots \\ \rho & 1 & ? & ? & \ldots \\ ? & ? & 1 & \rho & ? \\ ? & ? & \rho & 1 & ? \\ ? & ? & ? & ? & \ddots \end{bmatrix}$$

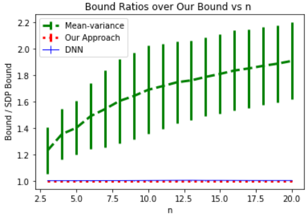
- 50 random instances
- Matlab-SDPT3 solver with YALMIP interface

Computations: Bounds

Table: Ratio of bounds over tight bound (Large-SDP) for various ρ values for $n = 6$. While the comonotone distribution is optimal under marginal information for the sum of waiting times objective (supermodular), the mean-variance bound is not necessarily tight for $\rho = 1$.

	Mean-variance			Our Approach			DNN Relaxation		
ρ	mean	min	max	mean	min	max	mean	min	max
-1.0	1.489	1.054	2.028	1	1	1	1.001	1	1.008
-0.7	1.251	1.036	1.492	1	1	1	1.001	1	1.006
-0.3	1.141	1.023	1.285	1	1	1	1.001	1	1.004
0.0	1.088	1.016	1.185	1	1	1	1.001	1.001	1.007
0.3	1.051	1.010	1.111	1	1	1	1.001	1	1.002
0.7	1.017	1.001	1.039	1	1	1	1.001	1	1.001
1.0	1.010	1	1.055	1	1	1	1.002	1	1.056





Computations: Execution time

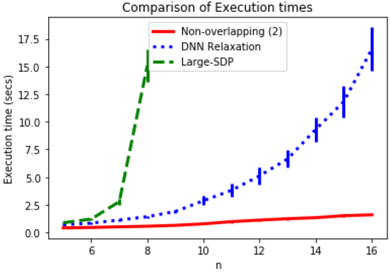


Figure: Execution times in seconds of various approaches with n

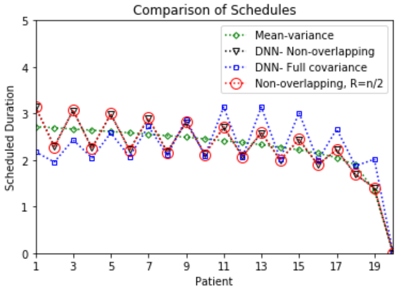
n	Mean	Min	Max
30	8.397	8.052	8.835
40	19.565	18.712	21.127
50	41.215	38.515	48.330
60	78.533	75.563	82.552
70	129.533	122.533	142.875
80	227.400	206.607	244.174
90	416.586	343.712	478.861
100	672.803	611.037	716.489

Table: Execution times (in sec) for solving the reduced semidefinite program

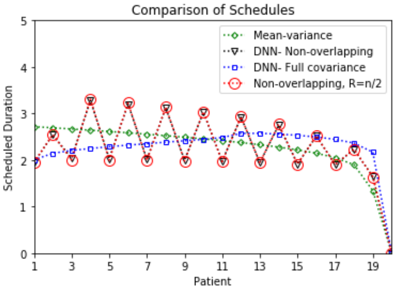
Computations: Optimal Schedules

- $n = 20$ patients
- $\mu_i = 2\ \forall i \in [n]$
- $\sigma_i = 0.5\ \forall i \in [n]$
- Vary correlation between consecutive patients $\rho \in \{1, 0, -0.5, -1\}$
- Feasible region of schedules $\sum_i s_i \leq 45, s_i \geq 0$
- Compare four approaches with mean and second moment information:
    - SOCP - Variance
    - DNN relaxation - Full covariance (set remaining correlations to 0)
    - DNN relaxation - Non-overlapping
    - Reduced SDP - Non-overlapping

Computations: Optimal Schedules

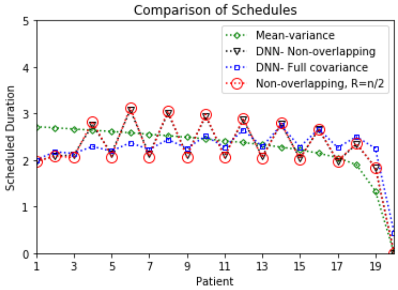


(a) Correlation between patient 1 and 2 = correlation between patients 3 and 4 = ... = $\rho = 1$. Mean-Variance bound = 25.6151, DNN relaxation (full covariance) bound = 15.9465, DNN relaxation (non-overlapping) bound = 25.1534, Reduced SDP (non-overlapping) bound = 25.0688

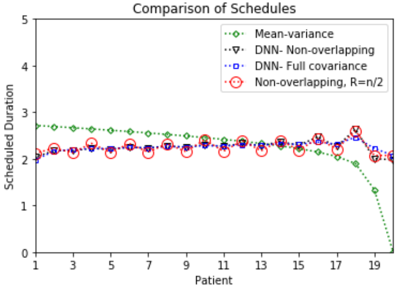


(b) Correlation between patient 1 and 2 = correlation between patients 3 and 4 = ... = $\rho = 0$. Mean-Variance bound = 25.6151, DNN relaxation (full covariance) bound = 11.4267, DNN relaxation (non-overlapping) bound = 19.8607, Reduced SDP (non-overlapping) bound = 19.7474

Computations: Optimal Schedules



(a) Correlations between patient 1 and 2 = correlations between patients 3 and 4 = ... = $\rho = -0.5$. Mean-Variance bound = 25.6151, DNN relaxation (full covariance) bound = 9.4195, DNN relaxation (non-overlapping) bound = 14.7904, Reduced SDP (non-overlapping) bound = 14.6842



(b) Correlation between patient 1 and 2 = correlation between patients 3 and 4 = ... = $\rho = -1$. Mean-Variance bound = 25.6151, DNN relaxation (full covariance) bound = 4.2223, DNN relaxation (non-overlapping) bound = 4.2290, Reduced SDP (non-overlapping) bound = 4.1162

THANK YOU!