Analyzing Policies in Dynamic Robust Optimization

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Models and Algorithms for Sequential Decision Problems Under Uncertainty

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joint work with Dan Iancu (Stanford)

Two-stage Dynamic Robust Optimization

Problem: $\min_{\boldsymbol{x}} \max_{\boldsymbol{w} \in \mathcal{W}} \min_{\boldsymbol{y}} f(\boldsymbol{x}, \boldsymbol{w}, \boldsymbol{y})$

$\boldsymbol{x}$ chosen $\mapsto$ $\boldsymbol{w}$ revealed $\mapsto$ $\boldsymbol{y}$ chosen (in response to $\boldsymbol{w}$)

Two-stage Dynamic Robust Optimization

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x chosen $\quad \mapsto \quad$ w revealed $\quad \mapsto \quad$ y chosen (in response to w)

- The model can be solved via Dynamic Programming (DP):
  - Given x, w $\quad \rightarrow \quad$ find $y^\star(x, w)$ $\quad \rightarrow \quad$ find $x^\star$
- For most problems, the DP approach is not practical

Simple Policies/Decision Rules

$$\text{Problem: } \min_{\boldsymbol{x}} \max_{\boldsymbol{w} \in \mathcal{W}} \min_{\boldsymbol{y}} \; f(\boldsymbol{x}, \boldsymbol{w}, \boldsymbol{y})$$

- Restrict $\boldsymbol{y}$ to a simple function of $\boldsymbol{w}$ (instead of the optimal response)
- Static Decision Rule: fix $\boldsymbol{y}$ to be independent of $\boldsymbol{w}$
- Linear Decision Rule (aka affine policies): set $\boldsymbol{y} = Q\boldsymbol{w} + \mathbf{q}$
- Other decision rules include quadratic, piece-wise linear, finite adaptivity, etc

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Motivation of This Research

When are simple decision rules (near) optimal?

Literature Review

- Tractability and empirical performance for linear decision rules
  Charnes et al. [1958], Gartska and Wets [1974], Löfberg [2003], Ben-Tal et al. [2004], Ben-Tal et al. [2005], Shapiro and Nemirovski [2005], Chen et al. [2008], Skaf and Boyd [2008], Goh and Sim [2010], Kuhn et al. [2011], ...
- (Sub)-Optimality of static decision rules
  Ben-Tal et al. [2009], Bertsimas and Goyal [2010], Bertsimas et al. [2015], Marandi and den Hertog [2017]
- (Sub)-optimality of linear decision rules in robust optimization
  Ben-Tal et al. [2009], Bertsimas et al. [2009], Iancu et al. [2013], Bertsimas and Goyal [2012], Ardestani and Delage [2016], Simchi-Levi et al. [2016]
- Other decision rules in robust optimization
  Chen and Zhang [2009], Bertsimas and Caramanis [2010], Bertsimas et al. [2011a], Bertsimas et al. [2011b], Hanasusanto et al. [2015]

Talk Outline

1. Model and Basic Setup
2. Characterizing the Performances of Decision Rules
3. Optimality Conditions in Robust Models
   - Application to Discrete Convex Functions
4. Conclusions

Model for This Talk

Our model: $$\min_{\boldsymbol{x}} \max_{\boldsymbol{w}\in\mathcal{W}} \min_{\boldsymbol{y}} \mathrm{f}(\boldsymbol{x}, \boldsymbol{w}, \boldsymbol{y})$$

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Our model (omitting $\boldsymbol{x}$): $$\max_{\boldsymbol{w} \in \mathcal{W}} \min_{\boldsymbol{y}} \ f(\boldsymbol{w}, \boldsymbol{y}) := J^{\star}$$

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- Let the dimension of $\boldsymbol{w}$ be m and the dimension of $\boldsymbol{y}$ be n
- A policy $q(\boldsymbol{w})$ that maps from $\mathbb{R}^m$ to $\mathbb{R}^n$ is *worst-case optimal* if

$$\max_{\boldsymbol{w} \in \mathcal{W}} f\big(\boldsymbol{w}, q(\boldsymbol{w})\big) \leqslant J^\star.$$

Model for This Talk

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Questions

Is the particular class of policies $\mathcal{Q}$ worst-case optimal? That is,

$$\min_{q \in \mathcal{Q}} \max_{w \in \mathcal{W}} \; f(w, q(w)) = J^\star?$$

If not, what is the performance of the best policy in $\mathcal{Q}$ relative to $J^\star$?

Key Assumptions in This Talk

Assumption 1

1. *Both* $\mathcal{W}$ *and* $\mathcal{Q}$ *are convex sets, and* $\text{ext}(\mathcal{W}) = \{\boldsymbol{w}^1, \dots, \boldsymbol{w}^K\}$.
2. *The function* $f\big(\boldsymbol{w}, q(\boldsymbol{w})\big)$ *is quasi-convex on* $\mathcal{W}$ *for each fixed* $q \in \mathcal{Q}$*; that is, for each* $\lambda \in [0, 1]$, $\boldsymbol{w}, \boldsymbol{w}' \in \mathcal{W}$, *we have*

$$f\big(\lambda \boldsymbol{w} + (1-\lambda)\boldsymbol{w}', q(\lambda \boldsymbol{w} + (1-\lambda)\boldsymbol{w}')\big) \leqslant \max\{f(\boldsymbol{w}, q(\boldsymbol{w})), f(\boldsymbol{w}', q(\boldsymbol{w}'))\}.$$

3. *The function* $f\big(\boldsymbol{w}, q(\boldsymbol{w})\big)$ *is convex on* $\mathcal{Q}$ *for each fixed* $\boldsymbol{w} \in \mathcal{W}$*; that is, for each* $\lambda \in [0, 1]$, $q, q' \in \mathcal{Q}$ *we have*

$$f\big(\boldsymbol{w}, \lambda q(\boldsymbol{w}) + (1-\lambda) q'(\boldsymbol{w})\big) \leqslant \lambda f(\boldsymbol{w}, q(\boldsymbol{w})) + (1-\lambda) f(\boldsymbol{w}, q'(\boldsymbol{w})).$$

Examples Satisfying Assumption 1

(Two-stage) adjustable robust linear optimization:

$$\min_{\boldsymbol{x}\in\mathcal{X}} \mathbf{c}^\mathsf{T}\boldsymbol{x} + \max_{\boldsymbol{w}\in\mathcal{W}} \min_{\boldsymbol{y}} f(\boldsymbol{x},\boldsymbol{w},\boldsymbol{y}), \quad \text{where } \mathcal{X},\ \mathcal{W} \text{ are linear polytopes}$$

$$\text{where } f(\boldsymbol{x},\boldsymbol{w},\boldsymbol{y}) = \begin{cases} \mathbf{d}^\mathsf{T}\boldsymbol{y} & \text{if } A\boldsymbol{x} + B\boldsymbol{y} \geqslant C\boldsymbol{w}, \\ +\infty & \text{otherwise.} \end{cases}$$

Assumption 1 is satisfied when $\mathcal{Q}$ is the set of linear decision rules.

Multi-stage robust inventory management:

$$\min_{x} \min_{y_1} \Big(c_1(y_1, x) + \max_{d_1\in\mathcal{D}_1} \Big(h_1(I_2, x) + \ldots + \min_{y_T} \Big(c_T(y_T, x) + \max_{d_T\in\mathcal{D}_T} h_T(I_{T+1}, x) \Big) \ldots \Big) \Big)$$

$$\text{s.t. } I_{t+1} = I_t + y_t - d_t, \forall t \in \{1, 2, \ldots, T\}.$$

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(Two-stage) adjustable robust optimization with linear-fractional objective:

$$\min_{\boldsymbol{x}\in\mathcal{X}} \mathbf{c}^\top \boldsymbol{x} + \max_{\boldsymbol{w}\in\mathcal{W}} \min_{\boldsymbol{y}} f(\boldsymbol{x},\boldsymbol{w},\boldsymbol{y}), \quad \text{where } \mathcal{X},\ \mathcal{W} \text{ are linear polytopes}$$

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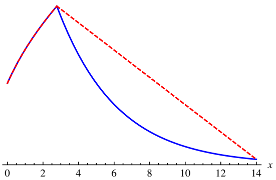

Example 1

Concave envelope of a 1D function

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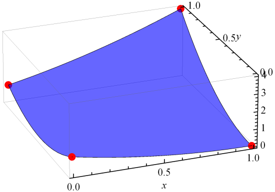

Example 2

Concave envelope of a 2D function

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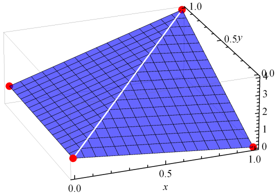

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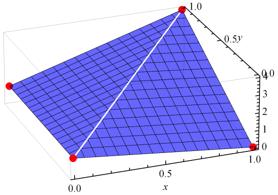

Example 2

When g is convex, $\mathrm{conc}(\mathrm{g})$ coincides with $\mathrm{conc}(\mathrm{g}^{\mathrm{ext}(\mathcal{W})})$

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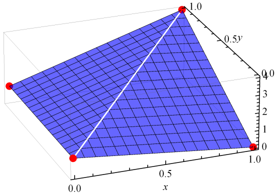

Example 2

When g is quasi-convex, $\mathrm{conc}(\mathrm{g}^{\mathrm{ext}(\mathcal{W})})$ preserves the maximum

Characterzing the Performance of $\mathcal{Q}$

Proposition 1

For each policy $\mathrm{q} \in \mathcal{Q}$*, define* $\mathrm{f}_{\mathrm{q}}(\cdot)$ *to be the function such that* $\mathrm{f}_{\mathrm{q}}(\boldsymbol{w}) = \mathrm{f}\big(\boldsymbol{w}, \mathrm{q}(\boldsymbol{w})\big)$*. Under Assumption 1, we have*

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Proof.

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&= \max_{\boldsymbol{w}\in\mathcal{W}} \min_{q\in\mathcal{Q}} \mathrm{conc}(f_q^{\mathrm{ext}(\mathcal{W})})(\boldsymbol{w}). && \text{(Sion's minimax)}
\end{aligned}$$

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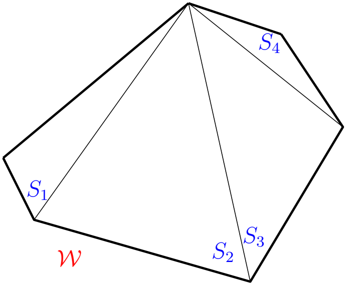
Concave Envelope Optimality Condition

Concave Envelope Optimality Condition

There is a worst-case optimal policy in $\mathcal{Q}$ if there exists a finite collection of convex sets $\{S_i\}_{i\in I}$ such that $\cup_{i\in I} S_i = \mathcal{W}$ and policies $\{q_i \in \mathcal{Q}\}_{i\in I}$ so that for each $i \in I$:

$$\mathrm{conc}(f_{q_i}^{\mathrm{ext}(\mathcal{W})})(\boldsymbol{w}) \leqslant J^*, \forall \boldsymbol{w} \in S_i.$$



Graphical Illustration

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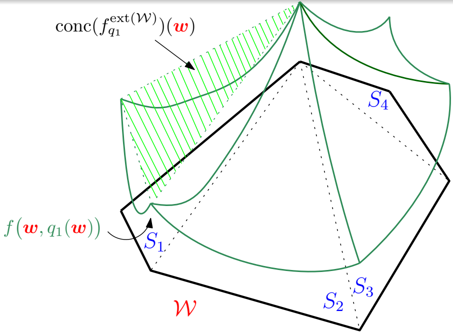

Graphical Illustration

Applying the Optimality Condition

Applying the Optimality Condition

Under Assumption 1, when can we precisely characterize the concave envelope from $\text{ext}(\mathcal{W})$?

- When the objective values at $\text{ext}(\mathcal{W})$ exhibits discrete convexity structures
- When the objective is linear (often occurs in dynamic robust LPs)

Supermodular Function and Lovász extension

Definition

Function $g : \mathbb{Z}^n \to \mathbb{R}$ *is supermodular if*

$$g(\max(x', x'')) + g(\min(x', x'')) \geqslant g(x') + g(x''), \ \forall x', x'' \in \mathbb{Z}^n.$$

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Kuhn triangulation: If $\text{ext}(\mathcal{W})$ is a sub-lattice of $\{0, 1\}^n$, then $\mathcal{W}$ can be partitioned into simplicies for the following form:

$$\Delta_\pi := \{\boldsymbol{w} \mid 0 \leqslant w_{\pi(1)} \leqslant \ldots \leqslant w_{\pi(n)} \leqslant 1\}, \ \pi \, \text{permutation of } [n]$$

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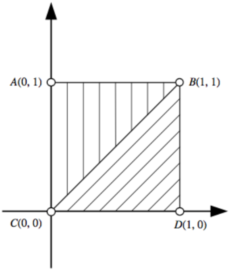

Kuhn triangulation for $\mathcal{W} = [0, 1]^2$

Relevant simplicies:

$$\text{conv}\Big(\{(0,0),(0,1),(1,1)\}\Big)$$

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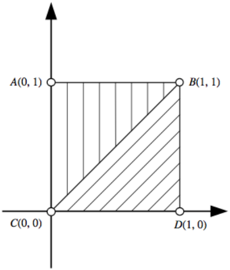

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If g is supermodular, $\text{conc}(g)$ (over $\mathcal{W}$) is a piece-wise linear interpolation on the extreme points of the Kuhn triangulation [Lovász, 1983].

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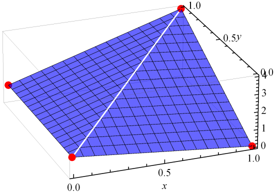
Function $g : \mathbb{Z}^n \to \mathbb{R}$ *is supermodular if*

$$g(\max(x', x'')) + g(\min(x', x'')) \geqslant g(x') + g(x''), \ \forall x', x'' \in \mathbb{Z}^n.$$



Kuhn triangulation for $\mathcal{W} = [0, 1]^2$

Relevant simplicies:

$$\mathrm{conv}\Big(\{(0,0),(0,1),(1,1)\}\Big)$$

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If g is supermodular, $\mathrm{conc}(g)$ (over $\mathcal{W}$) is a piece-wise linear interpolation on the extreme points of the Kuhn triangulation [Lovász, 1983].

Optimality with Supermodularity

Corollary (Worst-case Optimality with Supermodular Objective)

There exists a wost-case optimal policy $q \in \mathcal{Q}$ *if*

1. *Assumption 1 is satisfied.*
2. *The set* $\text{ext}(\mathcal{W})$ *is an integer sublattice of* $\{0, 1\}^n$.
3. *For each simplex* S_i *in the Kuhn triangulation, there exists* $q_i \in \mathcal{Q}$ *where* $f(\boldsymbol{w}, q_i(\boldsymbol{w})) \leqslant J^*$ *for each* $\boldsymbol{w} \in \text{ext}(S_i)$ *and* $f(\boldsymbol{w}, q_i(\boldsymbol{w}))$ *is supermodular on* $\mathcal{W}$.

Generalizes Theorem 1 of Iancu et al. [2013]:

- More general function (quasi-convex instead of convex in $\boldsymbol{w}$)
- Relax the condition on the objective function under the Bellman optimal response
- More general class of policies (no longer restricted to linear decision rules)

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Optimality with $L^\natural$-concavity

Corollary (Worst-case Optimality with $L^\natural$-concave Objective)

There exists a wost-case optimal policy $q \in \mathcal{Q}$ *if*

1. *Assumption 1 is satisfied.*
2. $\text{ext}(\mathcal{W}) \subset \mathbb{Z}^n$ *forms an* $L^\natural$*-convex set, and* $\{\Delta_{\pi^i, z^i}\}_{i \in I}$ *subdivides* $\mathcal{W}$.
3. *For each* i*, there exists* $q_i \in \mathcal{Q}$ *where* $f(\boldsymbol{w}, q_i(\boldsymbol{w})) \leqslant J^*$ *for each* $\boldsymbol{w} \in \text{ext}(\Delta_{\pi^i, z^i})$ *and* $f(\boldsymbol{w}, q_i(\boldsymbol{w}))$ *is* $L^\natural$*-concave on* $\text{ext}(\mathcal{W})$.

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- More general uncertainty set compared to the previous corollary (at the cost of more restrictive objective function)

Another Optimality Condition

Theorem 1

Under Assumption 1, the following are equivalent:

1. *There exists a wost-case optimal policy* $q \in \mathcal{Q}$.
2. *There exists a finite collection of convex sets* $\{S_i\}_{i \in I}$ *such that* $\cup_{i \in I} S_i = \mathcal{W}$ *and policies* $\{q_i \in \mathcal{Q}\}_{i \in I}$ *so that for each* $i \in I$*:*

$$\mathrm{conc}(f_{q_i}^{\mathrm{ext}(\mathcal{W})})(\boldsymbol{w}) \leqslant J^*, \forall \boldsymbol{w} \in S_i.$$

3. *There exists* $\hat{\boldsymbol{w}} \in \mathcal{W}$*, a finite collection of convex sets* $\{S_i\}_{i \in I}$*, policies* $\{q_i \in \mathcal{Q}\}_{i \in I}$*, and vectors* $\{\mathbf{g}_i\}_{i \in I}$ *such that:*

$$\begin{aligned} \hat{\boldsymbol{w}} \in S_i, \forall i \in I, \ \textit{and } \mathcal{W} - \hat{\boldsymbol{w}} \subset \mathrm{cone}(\cup_{i \in I} S_i), \\ f(\hat{\boldsymbol{w}}, q_i(\hat{\boldsymbol{w}})) \leqslant J^*, \forall i \in I, \\ f\big(\boldsymbol{w}, q_i(\boldsymbol{w})\big) \leqslant (\boldsymbol{w} - \hat{\boldsymbol{w}})\mathbf{g}_i + f(\hat{\boldsymbol{w}}, q_i(\hat{\boldsymbol{w}})), \ \forall i \in I, \boldsymbol{w} \in \mathrm{ext}(\mathcal{W}), \\ \mathbf{s}^{\mathsf{T}} \mathbf{g}_i \leqslant 0, \ \forall \, \mathbf{s} \in S_i. \end{aligned}$$

Connetion to Integerality Gap of an Integer Program

Recall that $\text{ext}(\mathcal{W}) = \{\boldsymbol{w}^1, \ldots, \boldsymbol{w}^K\}$, consider the optimization problem:

$$\begin{aligned} \max_{t,\boldsymbol{\lambda}} \ & t \\ \text{s.t.} \quad & t \leqslant \sum_{j=1}^{K} \lambda_j f(\boldsymbol{w}^j, q(\boldsymbol{w}^j)), \ \forall\, q \in \mathcal{Q}, \\ & \sum_{j=1}^{K} \lambda_j = 1,\ \lambda_j \in \{0,1\},\ \forall 1 \leqslant j \leqslant K. \end{aligned} \tag{IP}$$

Corollary

Suppose that Assumption 1 holds and $\mathcal{Q}$ contains all static decision rules. Then

$$\min_{q\in\mathcal{Q}} \max_{\boldsymbol{w}\in\mathcal{W}} f\big(\boldsymbol{w}, q(\boldsymbol{w})\big) - J^* \leqslant \textit{Integrality Gap of (IP)}.$$

Conclusions

- A general theory for studying the performance of simple decision rules in dynamic robust optimization
- Characterization of policy performances through concave envelopes
  - The approach using minimax in dynamic robust optimization problems deserve more attention in the literature
- Optimality of (affine) policies using concave envelopes and discrete convexity
- Optimality and sub-optimality guarantees of static policies for two-stage robust linear programs

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THANK YOU!

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