Localized radial basis function methods for PDEs in thin volumes

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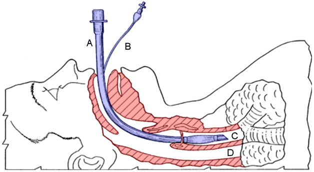
Thin PDEs

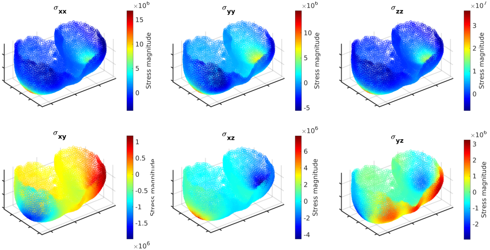
The medical problem



- Mechanical ventilation of intensive care patients leads to *Ventilator induced diaphragmatic dysfunction* (VIDD).
- In normal breathing the diaphragm contracts, but ventilation stretches the muscle.
- Significant loss off function in short time, $\mathcal{O}(24h)$.
- Goal: Understand effect, reduce effect, plan rehabilitation.

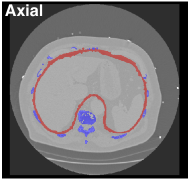
The numerical problem for the diaphragm

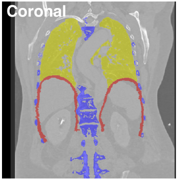
- Non-trivial patient-specific geometry. Thin domain with aspect ratio 1:100. *Resolution?*
- Mix of pressure boundary conditions and attachment conditions. *Good normals needed!*
- Non-linear constitutive relations and large deformation.
- Coupled to rib rotation, abdomen and lungs.

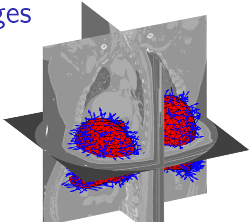


The input data

Manually segmented medical images

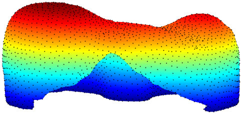


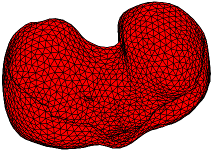




Low contrast leads to noisy segmentation. The image normals are not useful for computational purposes.

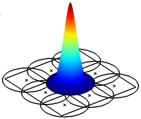
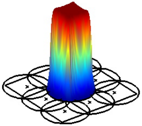
Conversion to point cloud/tetrahedral mesh



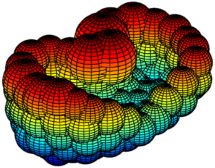
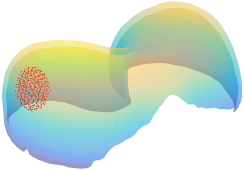


Noise both in surface representation and normals. Very few nodes inside the volume. Mesh artifacts.

Localized radial basis function methods

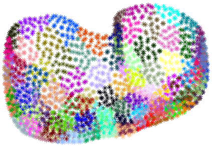
- Meshfree methods working on scattered nodes.
- Approximations $\tilde{u}(x) = \sum_{j=1}^{N} \lambda_j \phi(\|x - x_j\|)$.
- RBF-PUM: Approximations on patches are combined using weight functions $\tilde{u}(x) = \sum_{k=1}^{M} w_k(x)\tilde{u}_k(x)$.
- RBF-FD: Stencil weights are computed for each node using the local RBF approximation.

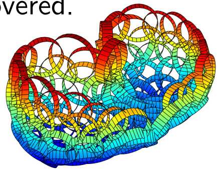



Geometry representation—Patches

Assumptions and needs

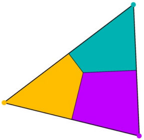
- The initial point cloud is quasi uniform.
- There is some kind of surface representation.
- Patches are convex objects.
- There is only one layer of patches.
- Each part of the domain is covered.





Method

1. Use `kmeans` for patch centers and PCA for patch orientation.
2. Extend radius to cover each surface element and to ensure overlap.



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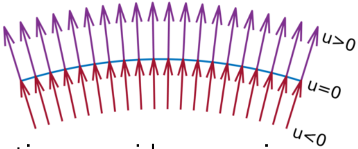
Smoothing surface approximation

- In the local coordinate system of each patch, we construct the functions $z_{\text{in}}(x, y)$ and $z_{\text{out}}(x, y)$.
- The same node template is used for all patches.
- A least squares approximation of the initial surface points is computed.
- We use a polyharmonic spline basis $\phi(r) = |r|^3$ augmented with a linear polynomial basis.
- The Woodbury formula is used to exclude nodes outside the data support.

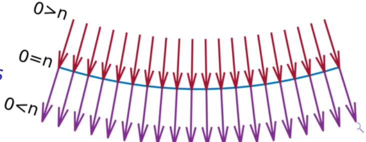
No obvious way to add surfaces.

Global distance function



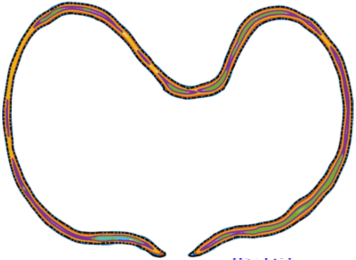
- The two surface approximations provide non-noisy 3-D distance data. Opposite directions.
- We again use least squares spline approximation on template nodes, now within the 3-D cylinder patch.
- The local function value in patch Ω_j is given by $u_j = \max(u_{\text{in}}, u_{\text{out}})$.
- The global function is constructed as $U = \sum_{j=1}^{M} w_j u_j$, where w_j and u_j are C^2 at the surface.
- We can use U to compute normals and to place node points.

Piret (2012) Orthogonal gradients

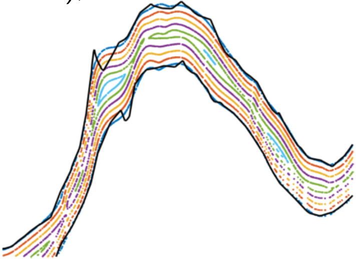


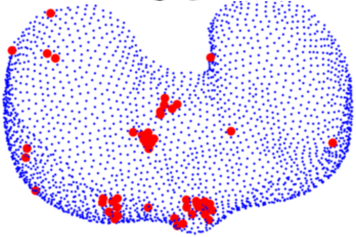
Evaluating the geometry representation

Mesh intersection (black), smooth contours.



Locally non-smooth mesh (black), smooth contours.





Inner surface, trouble spots.

Diff [mm]	Inner	Outer
Max	2.6	2.4
Min	-3.3	-2.4
Mean	0.002	0.011
RMS	0.40	0.47
Relative	$\sim$ 4%	

Anisotropic PDE approximation

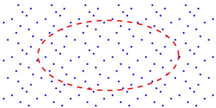
Why do we need it?

- There are relevant changes in the thickness direction.
- 'Long' dimensions are expensive to (over)resolve.

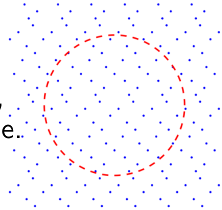
How do we do it?

- RBF-FD: Adjust nodes and stencils.
- RBF-PUM: 'Blow up' local thickness.

$h_x = 10h_y$



With scale invariant basis, this is the same.



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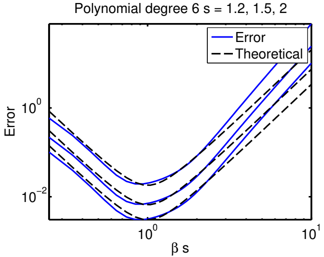
Approximation analysis + experiment

- Polyharmonic splines + poly = scale invariant.
- PHS + poly of degree $k \Rightarrow$ Exact for polynomials of degree $\leq k$.
- For 'well distributed' nodes, only the number of nodes in the local approximation are important.
- By splitting the function into a polynomial part and a remainder, we get $\|e\|_{L_2(\Omega)} \leq Ch^{k+1}|f|_{W_2^{k+1}(\Omega)}$.
- Scaling arguments: Best stencil makes f 'round'.

Anisotropicity of function s and coordinate transformation β.

Interpolation experiment with anisotropic function and anisotropic stencil scaling. ($k = 6$ is the best picture.)



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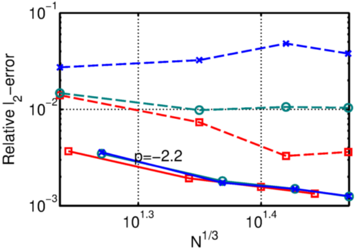
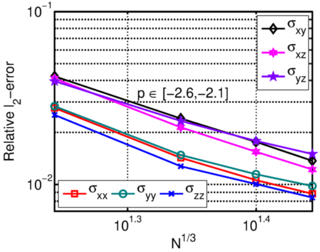
Preliminary results linear elasticity

Displacement



Poly degree $k = 3$ (red), $k = 4$ (green), $k = 5$ (blue).

Stresses



Poly degree $k = 3$ (all)

- Dashed lines, no refinment in the thickness direction.
- Done before the geometry part was finished.
- Stress convergence too good in comparison.

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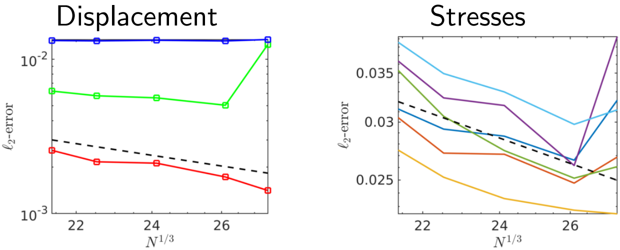
New experiment linear elasticity



Poly degree $k = 3$ (red), $k = 4$ (green), $k = 5$ (blue)
Trendline $p = -2$

Poly degree $k = 3$ (all)
Trendline $p = -1$

- More points in thickness needed for higher k.
- New geometry used, results are more consistent.
- For degree k and PDE of degree 2, order $k + 1 - 2$ is expected.

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Summary

- Framework for smoothing the geometry in place.
- RBF-PUM framework also in place.

ToDo

- Try out least squares RBF-PUM with local inflation for the PDE problem.
- Generating better node sets for RBF-FD.
- Some remaining geometrical details to handle.
- See if we can make convergence experiments and convergence analysis meet.