Determining a Lorentzian metric from the source-to-solution map for the relativistic Boltzmann equation

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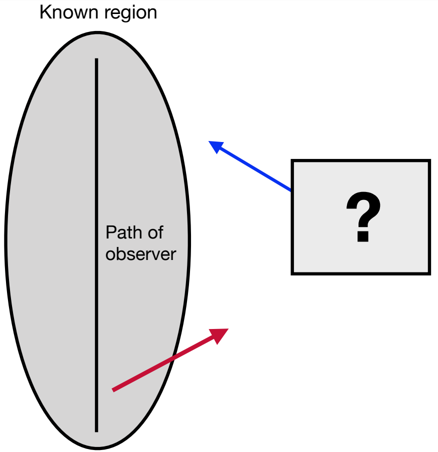
Outline

1. Recovering a Lorentzian metric from particle collisions
2. Results for wave signals
3. Result for particle signals
4. Main result proof sketch
5. Summary

Recovering a Lorentzian metric from particle collisions

Active measurement inverse problem

Question: Can you determine the shape of regions of spacetime from sending signals and measuring the resulting signals from interactions in the unknown region?



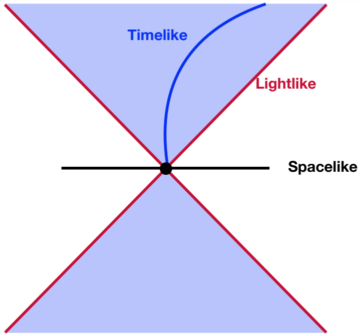
Setting

Known and unknown domains V and W lie in a time-oriented Lorentzian spacetime (M, g) with $\dim(M) \geq 3$.

- Lorentzian:
  - g has signature $(- + + \cdots +)$.
  - e.g. 4D Minkowski: $g = -dt^2 + dx^2 + dy^2 + dz^2$.
- Time oriented: tangent vectors $X \neq 0$ to M classified causally:
  - Timelike: $g(X, X) < 0$.
  - Lightlike: $g(X, X) = 0$.
  - Spatial: $g(X, X) > 0$.

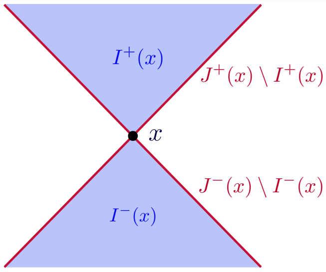
Relativistic setting

- Curves $\mu : [a, b] \to M$ classified causally:
  - Timelike: $g(\dot{\mu}(t), \dot{\mu}(t)) < 0$
  - Lightlike: $g(\dot{\mu}(t), \dot{\mu}(t)) = 0$
  - Spatial: $g(\dot{\mu}(t), \dot{\mu}(t)) > 0$



Setting

- (M, g) is time oriented: classify time/lightlike vectors/curves/regions as future $(+)$ or past $(-)$.
- Causal future/past of $x \in M$: $J^{\pm}(x) = \{y \in M : \exists$ lightlike or timelike geodesic from x to $y\}$.
- Chronological future of $x \in M$: $I^{\pm}(x) = \{y \in M : \exists$ timelike geodesic from x to $y\}$.
- Future/past light cone of $x \in M$: $L^{\pm}(x) = J^{\pm}(x) \setminus I^{\pm}(x)$.



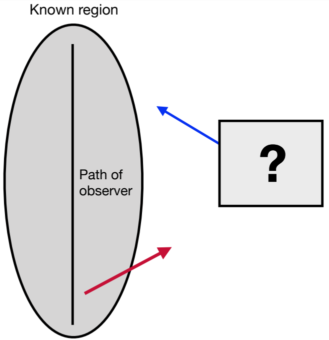
Setting

Global hyperbolicity:

- No closed causal paths in (M, g).
- $x, y \in M$, $x < y$, $J^+(x) \cap J^-(y)$ is compact.
- Implies $M = \mathbb{R} \times N$, $(\{t\} \times N, g|_{\{t\}\times N})$ Riemannian.

Setting

Goal: **Interactions occur in (M, g). Send and measure signals on $(V, g|_V)$. Use this to recover g on unknown domain $W \subset M$.**



Results for wave signals

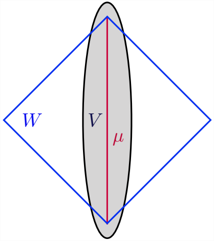
Wave signals

- Belishev-Kurylev (1992): N has boundary, $g|_{\{t\}\times N}$ independent of t, knowledge of Dirichlet-to-Neumann map for $\Box_g u = 0$ determines g. Tataru (1995): extended to g analytically depends on t.
- Kurylev-Lassas-Uhlmann (2017): If $\dim(M) = 4$, $V \subset M$ a known open neighbourhood of a timelike path $\mu : [-1, 1] \to M$, then the data

$$(V, g|_V) \text{ and } L_V : f \mapsto u|_V$$

where $\Box_g u + au^2 = f$, $u|_{t<0} = 0$, $f \in C_0^6(V)$, $||f||_{C_0^6(V)} < \epsilon$, determines $W = I^-(\mu(1)) \cap I^+(\mu(-1))$ and $g|_W$ up to conformal factor.

Kurylev-Lassas-Uhlmann (2017)



- The result of K-L-U used the nonlinearity au^2 as a tool with which to gain information.
- The nonlinearity dictates the interaction of the waves.
- Used microlocal techniques to show the interaction of 4 waves produced a point source spherical wave.
- Showed that you can determine the earliest time which you observe such a wave in V.

Extended work

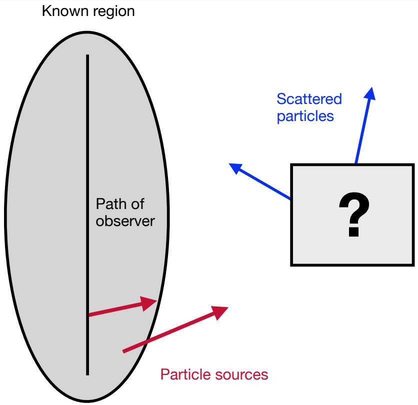
Using similar techniques:

- Wang-Zhou (2016), Lassas-Uhlmann-Wang (2016): Classes of semilinear wave equations.
- Lassas-Uhlmann-Wang (2017). Einstein-Maxwell equations.
- Kurylev-Lassas-Oksanen-Uhlmann (2018). Linear wave coupled with Einstein equation.

Result for particle signals

Active measurement inverse problem

Question: Can we determine a regions of spacetime from sending sources of particles and measuring the emitted light from the particle collisions?



Particle kinematics

- In the absence of forces, particles should travel along timelike/lightlike geodesics.
- Phase space: for $U \subset M$,

$$\mathcal{P}_m(U) := \{(x,p) \in TU \,:\, g(p,p) = -m^2,\ p \text{ future-directed}\}$$
$$\overline{\mathcal{P}}(U) := \bigcup_{m \geq 0} \mathcal{P}_m(U).$$

- Particles: $u : \overline{\mathcal{P}}(U) \to (0,\infty)$. View as average ensemble of possible particle states.

Liouville-Vlasov equation:

$$\mathcal{X} u(x,p) = 0 \quad \text{on } \overline{\mathcal{P}}(U).$$

where $\mathcal{X} =$ Geodesic vector field.

Kinematics in the presence of collisions and sources

Relativistic Boltzmann equation:

$$\mathcal{X} u(x,p) = \mathcal{A}[u,u](x,p) + f(x,p) \quad \text{on } \overline{\mathcal{P}}(U)$$

Collision operator:

$$\begin{aligned}\mathcal{A}[u,v](x,p) = &\int_{\Sigma_{x,p}} A(x,p,q,p',q')u(x,p')v(x,p'')dV(p',q') \\ &- \int_{\Sigma_{x,p}} A(x,p,q,p',q')u(x,p)v(x,\tilde{p})dV(p',q')\end{aligned}$$

- $A : TM^4 \to [0,\infty)$ is the **shock cross-section**.
- Look at collisions conserving momentum: $\Sigma_{x,p} = \{(x,p,q,p',q') \in TM^4 \,:\, p+q = p'+q'\}$.

Relativistic Boltzmann Equation

- Gain term:

$$\mathcal{A}_{gain}[u,v](x,p) = \int_{\Sigma_{x,p}} A(x,p,q,p',q')u(x,p')v(x,p'')dV(p',q').$$

- Loss term:

$$\mathcal{A}_{loss}[u,v](x,p) = u(x,p)\int_{\Sigma_{x,p}} A(x,p,q,p',q')v(x,\tilde{p})dV(p',q').$$

$\mathcal{X}u(x,p) = \mathcal{A}[u,u](x,p) + f(x,p)$ describes behaviour of

- plasmas.
- particles such as electrons, protons, photons, ...
- Bose-Einstein condensates.
- quasiparticles.
- ect.

Our setting

- **Data:** send and measure particles from a known open set $V \subset M$.
- **Goal:** Use the information to determine unknown region $W \subset M$ and $g|_W$.
- Consider

$$\mathcal{X}u(x,p) = \mathcal{A}[u,u](x,p) + f(x,p) \quad \text{on } \overline{\mathcal{P}}((0,\infty)\times N)$$
$$u(x,p) = 0 \quad \text{for } x \in (-\infty,0)\times N.$$

- Data encoded in the **source-to-solution map**

$$\Phi_V : C_c^\infty(J^+(V)) \to \mathcal{D}'(L^+(V)), \quad f \mapsto u|_V,$$

where $u(x,p)=0$ for $x \in [0,\infty)\times N$, and $\mathcal{X}u = \mathcal{A}[u,u]+f$ on $\overline{\mathcal{P}}(0,\infty)\times N)$.

Relativistic Boltzmann Equation

Local existence of solutions to Boltzmann problem $\implies \Phi_V$ well-defined.

- Existence results for Boltzmann Cauchy problem are known for certain admissible kernels of $\mathcal{A}$.
- No clear idea of what is a "good" collision operator.
- K. Bichtler (1967): for globally hyperbolic spacetimes and certain bounds on $A(x, p, q, p', q')$ and exponentially decaying data.
- For $||\mathcal{A}[u, v]||_B \leq ||u||_B ||v||_B$:
  - D. Bancel (1973). Globally hyperbolic spacetimes.
  - H Andréasson (2005).

Admissible collision kernels

We say A is an **admissible collision kernel** if

1. $A \in C^\infty\left(\bigcup_{(x,p)} \Sigma_{(x,p)}\right)$.
2. There is a uniform $C > 0$ such that

$$\int_{\Sigma_{x,p}} A(x,p,q,p',q')dV(x,p;q,p',q') \leq C,$$

   for every $(x,p) \in \bar{\mathcal{P}}(M)$.
3. $A \geq 0$ and $A(x,0,\cdot,\cdot,\cdot) = 0$.
4. $\text{supp}(x \mapsto A(x,\cdot,\cdot,\cdot,\cdot))$ is compact.
5. $\exists$ lightlike and future-directed $p \in T_xM$ with $A(x,p,p'+q'-p,p',q) > 0$, $p',q' \in T_xM$ with $||p'||_g > 0$ and $||q'||_g > 0$.

Local existence for small data

Proposition (B, Kujanpää, Lassas, Liimatainen)

- *K be a compact set in $\overline{\mathcal{P}}((0,\infty)\times N)$.*
- *$A : \bigcup_{(x,p)} \Sigma_{(x,p)} \to \mathbb{R}$ be an admissible kernel.*

Then, there is an open set $\Omega \subset C^k_K(\overline{\mathcal{P}})$ with $0 \in \Omega$, such that if $f \in \Omega$,

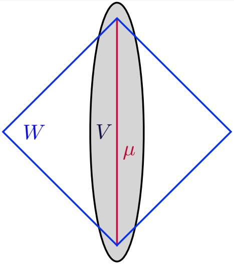
$$\mathcal{X}u(x,p) - \mathcal{A}[u,u](x,p) = f(x,p) \quad \textit{on} \quad \overline{\mathcal{P}}((0,\infty)\times N)$$

$$u(x,p) = 0 \quad \textit{on} \quad \overline{\mathcal{P}}((-\infty,0]\times N)$$

has a unique solution $u \in C(\overline{\mathcal{P}})$ with $||u||_{C^0(\overline{\mathcal{P}})} \leq C_K ||f||_{C^0(\overline{\mathcal{P}})}$ for some constant $C_K > 0$.

Theorem setup

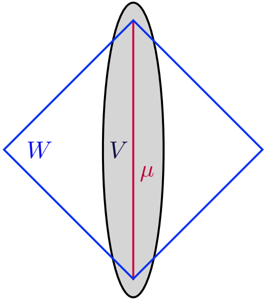
1. $(M = \mathbb{R} \times N, g)$ is a geodesically complete, globally hyperbolic, C^∞-smooth, Lorentzian manifold.
2. $\mu : [-1, 1] \to (0, \infty) \times N$ given smooth timelike curve.
3. Set $x^\pm := \mu(\pm 1)$.
4. There is an open neighbourhood $V \subset (0, \infty) \times N$ of μ such that $(V, g|_V)$ is known.
5. $\Phi_V : f \mapsto u|_V$ the source to solution operator for the Boltzmann equation, defined for f a neighbourhood of $0 \in C_c^\infty(J^+(V))$.



Main result

Theorem (B, Kujanpää, Lassas, Liimatainen)

For a given admissible scattering kernel A, the data $(V, g|_V)$ and the map Φ_V determines the metric g up to conformal class on the region $W := I^-(x^+) \cap I^+(x^-)$.



Main result proof sketch

Conormal distributions review

(Hörmander 18.2.8). Let $K \subset TM$ be a codimension k submanifold. We say that $u \in \mathcal{D}'(TM, \Omega^{\frac{1}{2}})$ is a **conormal distribution** to K of order $m \in \mathbb{R}$, denoted $I^m(K)$ if locally for

- $y \in \mathbb{R}^{2n+2}$ written $y = (y', y'')$, dual variable $\xi = (\xi', \xi'')$,
- $K = \{y' = 0\}$,

we have

- $u(x) = \int_{\mathbb{R}^k} e^{i\langle y', \xi' \rangle} a(y'', \xi')\, d\xi'$,

with symbol $\sigma(u) := a \in S^{m + \frac{n+1}{2} - \frac{k}{2}}(\mathbb{R}^{2n+2-k} \times \mathbb{R}^k)$.

Conormal distributions

- If $u \in I^m(K)$, then $\sigma(u) \in \mathcal{D}'(N^*K, \Omega^{\frac{1}{2}})$, where

$$N^*K := \{(y,\xi) \in T^*TM \,:\, y \in K,\, \langle \xi, \eta \rangle = 0,\, \forall \eta \in T_yK\}$$

  is the **conormal bundle** of K.
- In particular, $WF(u) \subset N^*K$.

Proof sketch

- Let $w_0 \in W := I^+(x^+) \cap I^-(x^-)$.
- Choose $\hat{x} \in V$.
- Let γ be the geodesic from $\hat{x}$ to w_0.
- We construct submanifolds $M_1 = \{(\hat{x}, \dot{\gamma}(0))\}, M_2 \subset \overline{\mathcal{P}}(V)$ with flowouts

  $$\Lambda_j = \{(x,p) \in TM \,:\, (x,p) = \dot{\gamma}_{(y,q)}(s),\, s \in \mathbb{R},\, (y,q) \in M_j\} \subset TM$$

  and projections $K_j \subset M$ satisfying
  - $\Lambda_1 \cap \Lambda_2 = \emptyset$
  - $K_1 \cap K_2 = \emptyset$.
- Construct sources $f_{j,\eta} \in I^m(N^*M_j)$, $j = 1,2$ which behave like delta distributions supported on M_j as $\eta \to 0$.

Proof sketch

- Consider the interaction

$$\mathcal{X} u_{\epsilon_1,\epsilon_2} = \mathcal{A}[u_{\epsilon_1,\epsilon_2}, u_{\epsilon_1,\epsilon_2}] + f_1\epsilon_1 + f_2\epsilon_2.$$

- We write $u_{\epsilon_1,\epsilon_2} := 0 + v_1\epsilon_1 + v_2\epsilon_2 + v_3\epsilon_1\epsilon_2 + R(\epsilon_1, \epsilon_2)$, where

$$\mathcal{X} v_j = f_j, \quad \mathcal{X} v_3 = \mathcal{A}[v_1, v_2] + \mathcal{A}[v_2, v_1].$$

- We show that Φ determines the source-to-solution map Φ^{2L} for the problem $\mathcal{X} v_3 = \mathcal{A}[v_1, v_2] + \mathcal{A}[v_2, v_1]$:

$$\Phi''(0; f_1, f_2) := \lim_{\epsilon\to 0} \frac{\Phi'(\epsilon f_2; f_1) - \Phi'(0; f_1)}{\epsilon} = \Phi^{2L}(0).$$

Proof sketch

- Consider the light-like signals received in V.
- Analyze the wavefront set of

$$v_3 = \lim_{\eta \to 0} \mathcal{X}^{-1} \left(\mathcal{A}[\mathcal{X}^{-1} f_{1,\eta}, \mathcal{X}^{-1} f_{2,\eta}] + \mathcal{A}[\mathcal{X}^{-1} f_{2,\eta}, \mathcal{X}^{-1} f_{1,\eta}] \right).$$

- We show the projection of $WF(v_3) = \cup_{b=0}^{B} T_{w_b} M$ for points $w_b \in W$.
- In particular, we determine the first observation of light from w_0 to $\hat{x}$.
- Kurylev-Lassas-Uhlmann (2017): This determines $(W, g|_W)$ up to conformal factor.

Summary

Summary

- **Question:** Can you determine regions of spacetime from sending particle signals and measuring the resulting light signals from interactions in the unknown region?
- **Yes!**
- Showed you can recover the structure of a causal diamond W from knowledge of the structure near the observer and the source-to-solution map for particle kinematics.
- **Key:** Nonlinear collision operator was the crucial element used to captured information about local structure of W.

Thanks!

Image Citations

- Brain scan: Wikimedia Commons/Sean Novak. (https://en.wikipedia.org/wiki/Magnetic_resonance_imaging_of_the_brain)
- Seismic: Grace Elton. (https://www.thinglink.com/scene/727582035165577217)
- VLT: ESO/A. Ghizzi Panizza (www.albertoghizzipanizza.com)
- Guide laser: ESO/G. Hüdepohl
- NASA / WMAP Science Team (http://map.gsfc.nasa.gov/media/121238/ilc_9yr_moll4096.png