$\mathbb{Q}$-curves over odd degree number fields

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Definitions

An isogeny (if no field is stated) is in this talk defined over $\overline{\mathbb{Q}}$.

An elliptic curve is called a $\mathbb{Q}$-*curve* if it is isogenous to all of its $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-conjugates.

If E/K is a $\mathbb{Q}$-curve, it is not necessarily isogenous over K to its conjugates.

$\mathbb{Q}$-curves in Diophantine equations

$\mathbb{Q}$-curves have been extensively used in the "modular method" to solving Fermat-type equations. It is often crucial to understand their Galois representations.

Ellenberg (2004) + Bennett, Ellenberg and Ng (2010): Solved about $A^4 + B^2 = C^p$ and $A^4 + 2B^2 = C^p$ using $\mathbb{Q}$-curves.

Dieulefait (2004): results about $x^4 + y^4 = z^p$ using $\mathbb{Q}$-curves

Dieulefait and Freitas (2011): Solved $x^5 + y^5 = 2z^p$ or $3z^p$ using $\mathbb{Q}$-curves

Bennett and Chen (2012): Solved $a^2 + b^6 = c^p$ or $3c^p$ using $\mathbb{Q}$-curves

Chen (2012): Results $a^2 - 2b^6 = c^p$ using $\mathbb{Q}$-curves

Bennett, Chen, Dahmen and Yazdani (2014): results about $a^3 + b^{3n} = c^2$ using $\mathbb{Q}$-curves

Why else do we care about $\mathbb{Q}$-curves

Ribet (2004) (assuming Serre's conjecture which was later proved): $\mathbb{Q}$-curves are exactly the elliptic curves over number fields that are modular, in the sense of being quotients of $J_1(N)$ for some N.

Which curves are $\mathbb{Q}$-curves

Any CM elliptic curve is a $\mathbb{Q}$-curve.

An elliptic curve defined over $\mathbb{Q}$ is a $\mathbb{Q}$-curve.

A base change of a $\mathbb{Q}$-curve is a $\mathbb{Q}$-curve.

A twist of a $\mathbb{Q}$-curve is a $\mathbb{Q}$-curve.

A curve that is isogenous to a $\mathbb{Q}$-curve is a $\mathbb{Q}$-curve.

An elliptic curve E with $j(E) \in \mathbb{Q}$ is a $\mathbb{Q}$-curve.

Let $\mathcal{E}$ be the set of all elliptic curves.

$$\mathcal{E} \supset \{\mathbb{Q} - \text{curves}\} \supset \{E \text{ isogenous to } E_1 \mid j(E_1) \in \mathbb{Q}\} \supset$$

$$\supset \{E \mid j(E) \in \mathbb{Q}\} \supset \{E/\mathbb{Q}\}$$

Also

$$\{\mathbb{Q} - \text{curves}\} \supset \{E \in \mathcal{E} \mid E \text{ has CM}\}.$$

Questions, questions

$$\begin{aligned}
\mathcal{QC} &:= \{\mathbb{Q} - \text{curves}\}\\
\mathcal{IJ} &:= \{E \text{ isogenous to } E_1 \mid j(E_1) \in \mathbb{Q}\}\\
\mathcal{J} &:= \{E \mid j(E) \in \mathbb{Q}\},\\
\mathcal{B} &:= \{E/\mathbb{Q}\},
\end{aligned}$$

Important tower of sets: $\mathcal{E} \supset \mathcal{QC} \supset \mathcal{IJ} \supset \mathcal{J} \supset \mathcal{B}$.

Which statements about Galois representaions of elliptic curves in each of these sets can we prove?

In particular are degrees of isogenies and sizes of torsion groups bounded?

I will not talk about CM elliptic curves. Their Galois representations are now well understood (Bourdon, Clark & collaborators, Lozano-Robledo).

Torsion groups over $\mathbb{Q}$

Our tower of sets: $\mathcal{E} \supset \mathcal{QC} \supset \mathcal{IJ} \supset \mathcal{J} \supset \mathcal{B}$.

For each of these sets S and for $d \in \mathbb{Z}_+$ denote by $S(d)$ the set of all such elliptic curves defined over all number fields of degree d.

$T(S) :=$ set of all possible torsion groups of elliptic curves in S.

Obviously $\mathcal{E}(1) = \mathcal{QC}(1) = \mathcal{IJ}(1) = \mathcal{J}(1) = \mathcal{B}(1)$.

Mazur (1977):
$T(\mathcal{E}(1)) = \{C_n : n = 1, \dots, 10, 12\} \cup \{C_2 \times C_{2m} : m = 1, \dots, 4\}$

Torsion groups over quadratic fields

Our tower of sets: $\mathcal{E} \supset \mathcal{QC} \supset \mathcal{IJ} \supset \mathcal{J} \supset \mathcal{B}$.

$T(\mathcal{E}(2)) = \{C_n : n = 1, \ldots, 16, 18\} \cup \{C_2 \times C_{2n} : n = 1, \ldots, 6\}$
$\cup \{C_3 \times C_{3n}, n = 1, 2\} \cup \{C_4 \times C_4\}$(Kenku,Momose '88,Kamienny '92).

$T(\mathcal{B}(2)) = T(\mathcal{E}(2)) \backslash \{C_n, n = 11, 13, 14, 18\}$. (N. (2014)).

$T(\mathcal{J}(2)) = T(\mathcal{B}(2)) \cup \{C_{13}\}$ (Tzortzakis (2018), Gužvić (2019)).

$T(\mathcal{QC}(2)) = T(\mathcal{J}(2)) \cup \{C_{14}, C_{18}\}$. (Le Fourn, N. (2018)).

Le Fourn (2013): over any imaginary quadratic field Serre's uniformity conjecture is true for curves in $\mathcal{QC} \backslash (\mathcal{IJ} \cup \mathcal{CM})$.

Where do torsion groups and isogenies appear?

Our tower of sets: $\mathcal{E} \supset \mathcal{QC} \supset \mathcal{IJ} \supset \mathcal{J} \supset \mathcal{B}$.

Where do elliptic curves over quadratic fields with certain torsion and isogenies appear?

Curves with C_{13} torsion are in $\mathcal{J} \backslash \mathcal{B}$. (Bosman, Bruin, Dujella, N. (2014))

Curves wit C_{18} torsion are in $\mathcal{QC} \backslash \mathcal{IJ}$. (Bosman, Bruin, Dujella, N. (2014))

Curves wit C_{16} torsion are in $\mathcal{B}$ (Bruin, N. (2016).)

For
$n = 22, 23, 26, 28, 29, 30, 31, 33, 35, 39, 40, 41, 46, 47, 48, 50, 59, 71$,
all curves with an n-isogeny, with finitely many explicitly stated exception, are in $\mathcal{QC} \backslash \mathcal{IJ}$ (Bruin, N. (2014)).

Torsion bounds over general number fields

Our tower of sets: $\mathcal{E} \supset \mathcal{QC} \supset \mathcal{IJ} \supset \mathcal{J} \supset \mathcal{B}$.

Order of groups in $T(\mathcal{E}(d))$ is bounded by some B_d. (Merel (1996))

Order of groups in $T(\mathcal{B}(d))$ for d not divisible by primes ≤ 7 is bounded by 16. (Gonzalez-Jimenez and N. (2016))

Order of groups in $T(\mathcal{J}(p))$, for p prime is bounded by 28. (Gužvić (2019))

Theorem (Cremona, N. (2020))

Order of groups in $T(\mathcal{QC}(p))$ for $p > 7$ prime is bounded by 16.

If one includes $p = 2, 3, 5, 7$ then the correct bound is almost certainly 28.

No such absolute bound can exist for $T(\mathcal{E}(d))$ when d runs through any infinite set of positive integers.

Isogeny bounds

Our tower of sets: $\mathcal{E} \supset \mathcal{QC} \supset \mathcal{IJ} \supset \mathcal{J} \supset \mathcal{B}$.

$I(S) :=$ set of all possible cyclic isogeny degrees of elliptic curves in S.

Note $I(\mathcal{J}(d)) = I(\mathcal{B}(d))$.

Mazur (1978) and Kenku (1980s) determined $I(\mathcal{B}(1))$.

N. (2015) - the largest prime in $I((\mathcal{IJ} \backslash \mathcal{CM})(d))$ is bounded by $3d - 1$ (and by $d - 1$ if we assume a weaker version of Serre's uniformity conjecture, which has been proven by Le Fourn and Lemos (2020)).

Isogeny bounds for $\mathbb{Q}$-curves

Theorem (Cremona, N. (2020))

Let $L = \{2, 3, 5, 7, 11, 13, 17, 37\}$.

a) *The primes in* $I((\mathcal{QC} \backslash \mathcal{CM})(d))$ *for odd* d *are contained in* L.

b) *If* d *is not divisible by any prime* $\ell \in L$, *then* $\max I((\mathcal{QC} \backslash \mathcal{CM})(d)) = 37$.

c) *For odd* d, $\max I(\mathcal{QC}(d)) \leq B_d$ *for some constant* B_d *depending only on* d.

$\mathbb{Q}$-classes

As the property of being a $\mathbb{Q}$-curve is twist and isogeny invariant, we see that it is a property of the $\overline{\mathbb{Q}}$-isogeny class.

We introduce the relation on $\overline{\mathbb{Q}}$ of $j_1 \sim j_2$ if elliptic curves E_1 and E_2 with $j(E_1) = j_1$ and $j(E_2) = j_2$ are in the same isogeny class.

This is an equivalence relation, so gives us a partition of $\overline{\mathbb{Q}}$ into isogeny classes.

We say that a class is a $\mathbb{Q}$-class if (all) elliptic curves in it are $\mathbb{Q}$-curves.

We say that a class is *rational* if it contains a $j \in \mathbb{Q}$.

Note that a rational class is automatically a $\mathbb{Q}$-class.

Structure of $\mathbb{Q}$-classes

To a pair of non-CM j_1, j_2 in the same class we define the *degree* $\deg(j_1, j_2)$ to be the degree of a cyclic isogeny between elliptic curves with those j-invariants.

We call an element j of a $\mathbb{Q}$-class a $\mathbb{Q}$-number.

The *degree* of a $\mathbb{Q}$-number j is the LCM of the degrees $\deg(j, g(j))$ for $g \in G_{\mathbb{Q}}$.

A $\mathbb{Q}$-number is *central* if it has square-free degree, in which case its Galois conjugacy class is called a *central (conjugacy) class*.

The existence of a central class in a $\mathbb{Q}$-class has first been proved by Elkies (1994).

Structure of $\mathbb{Q}$-classes

Theorem (Elkies $+\epsilon$)

Let $\mathcal{Q}$ be a non-CM $\mathbb{Q}$-class in $\overline{\mathbb{Q}}$. All central classes C in $\mathcal{Q}$ satisfy:

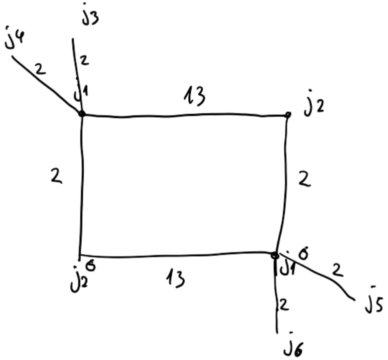
1. $|C| = 2^\rho$ *for some* $\rho \geq 0$;
2. $\mathbb{Q}(C)$ *is a polyquadratic field with* $\mathrm{Gal}(\mathbb{Q}(C)/\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^\rho$;
3. *the square-free degree* N *of one (and hence all)* $j \in C$ *has* r *prime factors, where* $r \geq \rho$ *and* $r = 0 \iff \rho = 0 \iff \mathcal{Q}$ *is rational.*

The quantities N, r and ρ, and the field $\mathbb{Q}(C)$, are the same for each central class in $\mathcal{Q}$, and we denote them $N(\mathcal{Q})$, $r(\mathcal{Q})$ and $\rho(\mathcal{Q})$ and $L_{\mathcal{Q}}$ respectively.

Open problem : how large can N, r and ρ be? Are they bounded?

Equivalently: when do quotients of $X_0(N)$ by groups of Atkin-Lehner involutions have non-cuspidal $\mathbb{Q}$-points.

Elliptic curves with 26-isogenies over quadratic fields



Here $j_1, j_2 \in K$, where K is a quadratic field, $\sigma \in G_{\mathbb{Q}}$ acts non-trivially on K. j_i for $i = 3, 4, 5, 6$ are defined over a quadratic extension L of K.

There are 2 central classes $C_1 = \{j_1, j_1^\sigma\}$ and $C_2 = \{j_2, j_2^\sigma\}$ and one non-central class $C_3 = \{j_3, j_4, j_5, j_6\}$.

We have $N = 26$, $\rho = 1$, $r = 2$.

Fields of defintion

Lemma

Let E_1/K_1 be isogenous to E_2/K_2. Then E_1 is isogenous to a twist of E_2 over K_1K_2.

Proposition

Let $j \in \mathcal{Q}$, $\mathcal{Q}$ non-CM. Then $L_{\mathcal{Q}} \subseteq \mathbb{Q}(j)$.

So $2^{\rho(\mathcal{Q})} \mid [\mathbb{Q}(j) : \mathbb{Q}]$.

An immediate corollary of the proposition is that any $\mathbb{Q}$-curve E/K is K-isogenous to a central curve, which is itself a base change of an elliptic curve over a polyquadratic field.

Removing the bar

Theorem (Elkies(1994))

Every non-CM $\mathbb{Q}$-curve over a number field K is $\overline{K}$-isogenous to an elliptic curve defined over a polyquadratic field.

Theorem (Cremona, N. (2020))

Every non-CM $\mathbb{Q}$-curve over a number field K is K-isogenous to an elliptic curve defined over a polyquadratic field.

This allows us to prove:

Theorem (The no-quadratic-subfields theorem)

If the non-CM $\mathbb{Q}$-class $\mathcal{Q}$ contains an element j such that $\mathbb{Q}(j)$ has no quadratic subfields, then $\mathcal{Q}$ is rational.

Proving our results

This means that for odd d we have $\mathcal{QC}(d) = \mathcal{IJ}(d)$ and the Galois representations of curves in $\mathcal{IJ}(d)$ are comparatively well understood and this allows us to obtain our results.

We also develop a quick algorithm which for an input of an elliptic curve quickly determines whether it is a $\mathbb{Q}$-curve or not.

The end

Thanks for listening!