Dynamics in Geometric Dispersive Equations and the Effects of Trapping, Scattering and Weak Turbulence II

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1 Overview of the Field

Of late, the primary advances in the field of dispersive PDE seem to be occurring in the study of models in Mathematical Physics that result in quasilinear equations, or at low regularity where semilinear equations take on more of a quasilinear structure. Many dispersive (and non-dispersive but related) PDE of this type arise naturally in applications, in such diverse areas as general relativity, plasma models, magnetics, optics, and water waves. These advances have opened doors to new models and new techniques, as well as strengthened the developing connections of the area to fluid dynamics. They have also provided pathways to connect to non-local operators through careful study of for instance the Dirichlet-to-Neumann map in fluids models, and opened up a host of questions regarding stability and dynamics in domains with an array of boundary conditions.

Moreover, some of the key physical examples of super-critical equations, such as the Einstein equations, and the gauge theories of particle physics, are inherently geometric. As a result, the field seems to be moving towards an inflection point, where we will see maximal advances on existence and stability theory for solutions to supercritical equations, quasilinear equation and/or equations very rough initial data with researchers moving strongly to find models and explore some of these challenging new directions.

This workshop brought together a group of people working in dispersive Partial Differential Equations five years beyond our first meeting. While many attendees were the same, we also had a host of new people come speak about progress in the field. A main take-away currently is that much progress is starting to be made in areas that have previously been somewhat unexplored due to the difficulties with understanding the equations. These include quasilinear models that include either complicated metrics or very low-regularity initial data for well-studied problems, detailed analysis of integrable systems, and blow-up dynamics in supercritical problems. As a result, complicated models in quantum field theories, fluids, geometric waves and more are becoming more tractable analytically.

2 Recent Developments and Open Problems

The workshop saw topics that could easily fit into a few developing areas: (i) development of quasilinear or degenerate dispersive models, (ii) stability in critical/super-critical models, (iii) Dispersive theory in fluids, (iv) applications of ideas from integrable systems to low-regularity studies of dispersive equations and/or stochastic PDE, (v) linear/nonlinear theory in domains and with variable coefficients. We will discuss the major topics in each category and summarize the results presented.

2.1 Quasilinear and Degenerate Schrödinger/Wave models

The workshop featured talks by Mihaela Ifrim, Jason Metcalfe, Sung-Jin Oh, and Benoit Pausader about solutions to nonlinear PDE in various settings where wave interactions take on a very different character on long time scales due to the lack of dispersion.

- Mihaela Ifrim - Low dimensional quasilinear systems of wave equations

  In joint work with A. Stingo [26], the authors study the system

$$(\partial_t^2 - \Delta)u = N_1(v, \partial v) + N_2(u, \partial v),$$
$$(\partial_t^2 - \Delta + 1)u = N_1(v, \partial u) + N_2(u, \partial u)$$

  where N_1 and N_2 are built to have classical quadratic null forms. For small data solutions of sufficient regularity, the authors are able to prove almost global well-posedness using a combination of vector field methods and local energy decay to establish pointwise decay estimates. These equations arise in the study of general relativity and are related to equations for the wave equation on wave-guides. In $3d$ and higher, these types of problems have been solved for some time using more classical techniques, but for low dimensions the weak dispersive decay makes the problem quite challenging. The use of local energy decay to get pointwise bounds for wave decay originated in the work of Tataru on Price's Law [48] and the dyadic version that appears here is due to Metcalfe-Tataru-Tohaneanu [40]. The spherical hyperbolic coordinates introduced by Tataru in [47] also play a major role. The authors are hoping to enhance the result to global by introducing the idea of testing by wave packets (see [24, 27, 25]), though this is work in progress. The take-away is that long time dynamics are possible for low-dimensional quasilinear wave/Klein-Gordon systems, but that to do so one needs very good energy estimates for the linearized system to get good pointwise bounds and even then the decay properties of the waves are much more subtle. However, the authors have now laid a foundation for these pursuits to be explored.

- Jason Metcalfe - Quasilinear Schrödinger equations

  In joint with with J.L. Marzuola and D. Tataru [38], the authors study local well-posedness for large data Quasilinear Schrödinger equations of the form

$$i\partial_t u + g^{jk}(u, \nabla u, \bar{u}, \bar{\nabla} u)\partial_j \partial_k u = F(u, \nabla u, \bar{u}, \bar{\nabla} u).$$

  Quasilinear Schrödinger models arise in various forms, but specifically in Density Functional Theory models for many body systems in electronic structure, plasma models, superfluid thin film models, in the study of rotating fluids and in the study of Hall/electron magnetohydrodynamics. The result establishes a robust short-time existence theorem the requires some care due to the metric dependence upon the solution in the dispsersion. In particular to pair with the author's small data results that relied upon a combination of energy estimates and local smoothing properties that were perturbative in the small data case, [37, 39], one must establish robust incoming estimates and non-trapping estimates that are stable under rough perturbations. This moves past prior work by Kenig-Ponce-Vega, Kenig-Ponce-Rolvung-Vega [31, 30] by defining translation invariant spaces and dramatically reducing regularity requirements. A paradifferential framework is introduced that simplifies some of the arguments from the authors previous works and provides a unifying framework for applications of the ideas contained within the result.

- Sung-Jin Oh - Degenerate Schrödinger equations from the Magnetic Hall Dynamics

  In joint work with I.J. Jeong (see [29] and forthcoming extensions), the authors study the equations

$$\partial_t u + u \cdot \nabla + \nabla p = \nu \Delta u + J \times B,$$
$$\partial_t B + \nabla \times E = 0,$$
$$\nabla \cdot u = 0, \ \nabla B = 0$$

with the assumptions that

$$J = \nabla \times B \ \text{(Ampere's Law)},$$
$$E + u \times B = \eta J + cJ \times B \ \text{(Ohm's Law \& Hall Current)}.$$

If $\eta = 0$, there is no resistivity and the model becomes a degenerate dispersion equation. Other degenerate dispersive models relating to hydrodynamic equations were studied recently in [17, 18]. The authors use the degeneracy to construct a strong ill-posedness result near trivial magnetic field backgrounds, but in forthcoming work will prove existence in a neighborhood of a non-trivial magnetic field.

- Benoit Pausader - Derivation of the Ion equation

  In joint work with Grenier, Guo and Suzuki [21], the authors derive a "Low Electron Mass Number" equation for ions built around the Euler-Poisson plasma model of interacting electrons and ions. The model is hydrodynamic again is is of the form

$$\partial_t \rho_i + \text{div}(\rho_i u_i) = 0,$$
$$\rho_i(\partial_t u_i + u_i \cdot \nabla u_i) + \nabla p_i + \rho_i \nabla \phi = 0,$$
$$-\Delta\phi + e^{\phi} = \rho_i.$$

  The original plasma model is a compressible two-fluid electrostatic model, but the ion equation looks somewhat incompressible. However, the ion equation is a singular limit of the two-fluid model, hence the convergence must be managed carefully. This is similar to the Euler equation as the limit of a 'low Mach number' compressible fluid. Energy estimates must be done very carefully to get uniform convergence in ϵ, and the resulting system is somewhat Klein-Gordon-like but with variable coefficients depending upon the ion equation solution. To get the correct energy estimates, one must use normal form analysis, as well as forms of local energy decay and equipartition of energy must be developed here that generalize the notion of the fluid momentum coordinates.

2.1.1 Related Open Problems

Large data blow-up for couple wave/Klein-Gordon quasilinear systems, non-flat geometries for coupled wave/Klein-Gordon systems, Global existence in low dimensions of wave/Klein-Gordon quasilinear systems, inclusion of Coulomb-like singular potentials in quasilinear Schrödinger models, improvements in well-posedness for quasilinear equations with Hamiltonian structure, ill-posedness and well-posedness for other degenerate Schrödinger models in physics, external magnetic fields or geometry in the ion equation, results and dynamics for problems without coercive energy quantities.

2.2 Stability in Critical/Super-critical equations

The workshop featured talks by Enno Lenzmann, Wilhelm Schlag and Birgit Schörkhuber about solutions to nonlinear PDE in various settings where wave interactions take on a very different character on long time scales due to the lack of dispersion.

- Enno Lenzmann - A new L^2 critical NLS equation In joint forthcoming work with Patrick Gerard, the authors introduce the equation

$$iu_t = -u_{xx} = 2(D_+(|u|^2))u,$$

  where $D_+ = \Pi_+ D$, $D = -i\partial_x$. The operator Π_+ is thus a projection onto positive frequencies and is related to the Szegő equation (see the recent survey of results in [16]). It is L^2 critical and has a stationary solution of the form

$$Q(x) = \frac{\sqrt{2}}{x+i},$$

  which can be seen to be unique in a sense up to symmetries. In ways, the equation has both defocusing and focusing tendencies, as it can be converted via a transformation to an equation of such a type. A Lax Pair can be constructed and the integrability of the model understood to some extent. However, many questions remain about the nature of the dynamics and behavior of solutions to this model.

- Wilhelm Schlag - Non-equivariant in/stability of critical wave maps

  In joint work with J. Krieger and C. Miao, the authors explore the stability of blow-up solutions near the equivariant 1 wave map in the non-equivariant class. The work is based upon extending the contstruction of blow-up solutions in [35] to the setting where the perturbations are not equivariant, and as a result extending the blow-up rigidity result of [34]. Weighted dispersive estimates are required of the form [15, 45, 46]. The functional equivalent of 'one-pass' theorems are required to ensure that the dynamics are not effected by energy that has radiated away to a certain extent.

- Birgit Schörkhuber - Self-similar blow-up solutions for super-critical wave equations

  In joint work with Glogić and Maliborski (see [20, 19]), the authors explore the stability of blow-up solutions constructed for supercritical wave equations in high dimensions. A key example is

$$\left(\partial_t^2 - \partial_r^2 - \frac{d-1}{r} \partial_r \right) u = u^3$$

  for $d \geq 5$. The solutions themselves are constructed explicitly, then studied in self-similar coordinates. A main issue is understanding the linear stability properties of the proposed states. Related models in include the symmetric version of Yang-Mills.

2.2.1 Related Open Problems

Other integrable models from Szegő type nonlinearities, Stability or blow-up solutions for wave maps with different targets and different domains, spectral properties of special solutions from the work of Schörkhuber et al, verification of the numerically observed dynamics and blow-up for supercritical waves.

2.3 Fluids

The workshop featured talks by about various aspects of fluids models by Albert Ai, Thomas Alazard, Roberto Camassa, Marcelo Disconzi, Jon Wilkening.

- Albert Ai and Thomas Alazard - Water Waves and Hele-Shaw equations

  The gravity wave equations have recently seen a great deal of progress as quasilinear techniques have become more readily available and the equations better understood. Through a combination of different methods, various groups have made progress on the problem recently, especially for gravity-waves in $2d$ on global time scales. Reports on this progress were made by Thomas Alazard and Albert Ai. The gravity-capillary wave equations can be represented as

$$\begin{cases} \partial_t h = |D|\,\psi + \{G(h)\psi - |D|\,\psi\} \\ \partial_t \psi = (\tau\Delta - g)\,h + \Bigg\{ \tau \left(\operatorname{div}\left(\dfrac{\nabla h}{\sqrt{1+|\nabla h|^2}} \right) - \Delta h \right) \\ \qquad\qquad -\dfrac{1}{2}|\nabla\psi|^2 + \dfrac{(G(h)\psi + \nabla h \cdot \nabla\psi)^2}{2\left(1+|\nabla h|^2\right)} \Bigg\}. \end{cases}$$

  for h the height of the fluid at the interface and ψ a trace of a related velocity field.

  Ai reported on recent work with Ifrim and Tataru in [1] where improvements to small data well-posedness for the gravity wave system ($\tau = 0$) are established. The key observation is the new energy estimate for the equations in holomorphic coordinates given by

$$\partial_t E^s \leq A_{\frac{1}{4}}^2 E^s,$$

  where $A_{\frac{1}{4}} \sim \| |D|^{\frac{1}{4}} \cdot \|_{BMO}$. This improved diagonalized energy estimate allows for a dramatic improvement of the local Cauchy theory and opens up the possibility of further improvement using refined Strichartz estimates as well. Very sharp energy estimates on the linearized paradifferential form of the equation are essential since one cannot apply normal form methods to the problem.

Alazard reported in improvements for the Dirichlet-to-Neumann map achieved with Lazar and at a similar time by Nguyen-Pausader [42, 3], as well as on identities and monotonic quantities for the Hele-Shaw flow. While not technically a dispersive equation, its evolution relies on the Dirichlet-to-Neumann map and can be related to key quantities in the so-called good unknown determination of the water wave problem, namely vertical, horizontal and time derivatives of the velocity potential restricted to the surface.

- Marcelo Disconzi - Strichartz estimates for relativistic fluids

  In joint work with C. Luo, G. Mazzone and J. Speck in [14], the authors introduce finite-time Strichartz estimates for compressible Euler equations with vorticity in order to construct local solutions. This can be written as a system of quasilinear wave equations, though to get low-regularity results one needs careful dispersive theory and control on the characteristic geometry. Generically, the transport terms make the terms non-perturbative. Vorticity is required for the argument by giving elliptic estimates for the transport variables. Hölder regularity is required on part of the data to make sense of the mapping properties.

- Roberto Camassa and Jon Wilkening - Water wave numerical and asymptotic models.

  These topics by applied mathematicians were about phenomenological fluids results relating to singularity formation in the hydrodynamic models for Euler equations in long-wave shallow water [12] and quasiperiodic waves in the gravity-capillary system [49]. In the first, self-similar solutions to asymptotic models are constructed for singularity formation in two-fluid systems with singular shock-like formation resulting from dry singularities. In the second, a zoo of special solutions to the gravity-capillary waves are computed numerically with remarkable structure, $u(\vec{k}x + \vec{\omega}t + \vec{z})$. Both results were supported by strong modeling and numerical calculations, and provide a rich landscape of theory questions to consider.

2.3.1 Related Open Problems

Analytic solutions from works of Camassa et al for generic construction of dry solutions and comparison to full Euler, existence and stability for the quasiperiodic waves of Wilkening et al, generalizations of existence times for small data from conformal coordinates to more general fluid domains and other formulations of Euler, long time dynamics in compressible Euler.

2.4 Integrable Systems, low-regularity and SPDEs

The workshop featured talks by about various aspects of wave equation models by Valeria Banica, Ben Harrop-Griffiths, Herbert Koch, Adrian Nachman and Hiro Oh.

- Valeria Banica - The binormal flow and singular initial data for cubic NLS

  In joint work with Luis Vega (see [6, 7, 8, 9, 10], motivated by regularly recurring polygons in the study of vortex rings, the consider finite dirac mass initial data for the cubic NLS in $1d$, which can be connected to the binormal flow via a transformation. Such low regularity creates complications with even defining the flow and must be done with great care.

- Ben Harrop-Griffiths - Low regularity NLS/mKdV

  In joint forthcoming work with Rowan Killip and Monica Visan based upon the work [33, 32], the authors construct a modified NLS flow built around the conserved quantities from the theory of complete integrability and use such a flow to construct solutions in H^s for $s > -\frac{1}{2}$. The key ideas relate to local smoothing estimates and tightness bounds to make the limits of the model flows converge to a solution of the actual flow.

- Herbert Koch - Multiple solitons in NLS

  In joint work with Tataru building off of [33], the authors construct multiple soliton solutions through the use of inverse scattering theory, and in the process observe stability and construct the soliton manifold for a large class of multi-soliton solutions that are close in the $H^{-\frac{1}{2}}$ norm to an N-soliton solution at time 0.

- Adrian Nachman- The Calderon Problem and the Davey-Stewartson system

  In joint work with Regev and Tataru in [41], the authors explore the invertibility of an inverse scattering transformation that both describes the evolution of the elliptic Davey-Stewartson system and also plays a major role in the reconstruction problem for the Calderon problem. The map

  $$Sq(k) = \frac{1}{2\pi i}\int_{\mathbb{R}^2} e_k(z)\overline{q(z)}(m_+(z,k) + m_-(z,k)dz$$

  where $m_\pm$ are solutions to

  $$\frac{\partial}{\partial \overline{z}} m_\pm = \pm e_{-k} q \overline{m_\pm}$$

  and

  $$e_k(z) = e^{i(zk+\overline{zk})}.$$

  serves as a nonlinear Fourier transform of sorts, and the authors prove it is an L^2 norm preserving map. The fundamental result is a beautiful exercise in harmonic analysis using maximal function estimates with far reaching implications in inverse problems.

- Tadahiro Oh - On the Stochastic Dispersive PDE

  In joint work with many authors, the speaker has explored well-posedness in a variety of stochastically forced dispersive models. Elements of renormalization theory must be used to make sense of the low-regularity of noise in the equation. See [43, 23, 22, 44] for some related results.

2.4.1 Related Open Problems

The fractional or non-linear Calderon problem and possible reconstruction algorithms, δ function regularity in NLS and the binormal flow (the Dirac comb solution), soliton gas in NLS, existence theory and Gibbs measures for stochastic models with other types of nonlinearities and noise.

2.5 Linear and Nonlinear Effects of Domains

The workshop featured talks by about various geometric aspects of dispersive equations by Zaher Hani, Nicolas Burq, Jonas Lührmann. Hani discussed kinetic equations for modeling frequency cascades NLS on the torus, Burq discussed boundary control theory in annuli and Luhrmann discussed 1d variable coefficient wave models.

- Zaher Hani - Kinetic equations in NLS

  In joint work with Yu Deng in [13], the authors derive a nonlocal equation known as wave kinetic equation from the dynamics of cubic NLS on large tori in certain scaling limits. These equations can then describe the statistics of how random waves behave the possibilities of cascades leading to weakly turbulent behaviors.

- Nicolas Burq - Boundary observability for the linear Schrödinger equation in annuli

  Based upon a paper by Anantharaman-Léautaud-Macia in [4] on the disc, Burq presented work on boundary observability for the Schrödinger equation on the annulus that builds off his previous works such as [11]. While he cannot get to the full disc, the results on the annulus follow from relatively well-understood tools involving semi-classical defect measures and positive commutators with commuting vectors. These control and observability results can generally be extended to nonlinear, small-data problems and relates as well to damped waves and other interesting geometric models, see [2, 5].

- Jonas Lührmann - Variable coefficient nonlinear $1d$ Klein-Gordon equations

  In joint work with Lindblad and Soffer in [36], the authors explore the equation

  $$(\partial_t^2 - \partial_x^2 + 1)u = (\alpha_0 + \alpha(x))u^2 + (\beta_0 + \beta(x))u^3,$$

  where serves as a model for the equation linearized around a kink solution in a nonlinear wave equation of the form

  $$(\partial_t^2 - \partial_x^2)\phi = -V'(\phi)$$

  with $V(x) \geq 0$ something like a double-well potential that is not decaying a 0. This type of problem can lead to the existence of kink solutions, and the goal is to develop a dispersive machinery for kind stability. So far, the authors can create a vector-field type approach to these problems using hyperbolic coordinates to prove pointwise decay in the cubic nonlinear case where $\alpha_0 + \alpha(x) = 0$. For the quadratic case, they have results for $\alpha_0 = 0$ depending upon the Fourier modes of $\hat{\alpha}$. Combining both results and getting to the actual kink problem requires future work, but the methods are robust. In any case, the decay is rather weak $t^{-\frac{1}{2}}$ since it is a $1d$ problem. See also recent work of [28] about kink-anti-kink pair rigidity and dynamics.

2.5.1 Related Open Problems

Classification, statistics and dynamics of weakly turbulent cascades in NLS, Nonlinear control and observability for NLS, Generalizations of control/observability with other metrics, Applications of control/observability to kink dynamics and stability in 1d Klein-Gordon.

3 Scientific Progress Made

Numerous collaborations and discussions took place between people in various sub-fields of PDE and numerical analysis. Many young people attended the conference and were thus able to learn a great deal about pioneering areas in the field. Several ongoing collaborations were strengthened at the same time. In the end the meeting was quite useful for bringing together many areas of dispersive PDE strongly impacted by geometry and nonlinearity and solidifying new highly nonlinear directions and applications in the field.

Participant list:
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