Non-central Funk-Radon Transforms: single and multiple

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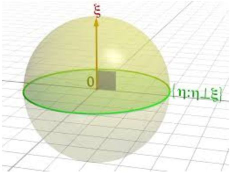
The Funk-Minkowski-Radon transform

Paul Funk (1911), based on a work by Minkowski (1904)

Integrates functions over the intersections of $S^{n-1} \subset \mathbb{R}^n$ and linear k-spaces E, $1 \leq k < n$ fixed:

$$(F_0 f)(E) = \int_{S^{n-1} \cap E} f(x) \, dA_{k-1}(x),$$

where dA_{k-1} is the surface area measure on $S^{k-1} = S^{n-1} \cap E$.



Thus, $F_0 : C(S^{n-1}) \to C(Gr_0(n,k))$ (functions on the k-Grassmanian), $Gr_0(n, n-1) \cong S^{n-1}$.

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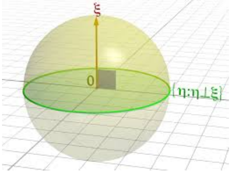

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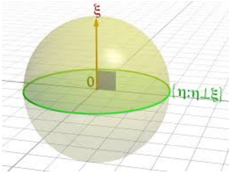

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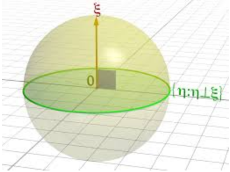

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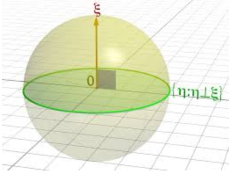

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Background

- Kernel = odd functions
- Injective on even functions

Inversion formula is written explicitly. E.g., *S.Helgason*, $n = 3, k = 2$:).

$$(F_+^{-1} g)(x) = \frac{1}{2\pi} \left[\frac{d}{ds} \int_0^\infty (F^* g)(\arccos\ v, x) v (s^2 - v^2)^{-\frac{1}{2}} dv \right]|_{s=1},$$

where

$$(F^* g)(p, x) = \frac{1}{2\pi cosp} \int_{|u|=1, \langle x,u\rangle = sin\ p} g(u)\ du.$$

It provides the right inverse operator:

$$F_0 F_0^{-1} f = f,\ f \in C(Gr_0(n, k)).$$

Action from the left:

$$F_0^{-1} F_0 f = f^+, f \in C(S^{n-1}),$$

where $f^+(x) = \frac{1}{2}(f(x) + f(-x))$- the even part of f.

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- Convex geometry, intersection bodies problems:

Volume of a k-dim linear cross-section is the Funk transform of the radial function:

$$V(K \cap P_k) = \int_{S^{n-1} \cap P} \rho_K^{k-1}(\theta) d\theta,$$

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Shifted Funk Transform: definition

Recent years: study Funk-type transform centered *not* at 0, which *integrates over non-central cross-sections*.

Definition

Let $a \in \mathbb{R}^n$. The (shifted) Funk transform centered at a is defined on $f \in C(S^{n-1})$ by

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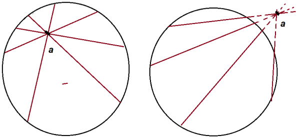

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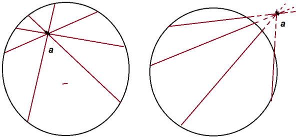

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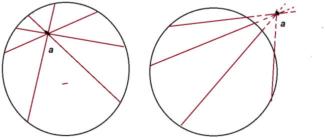

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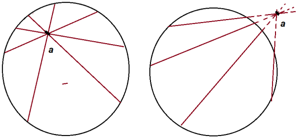

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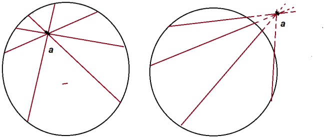

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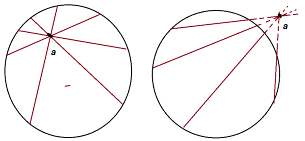

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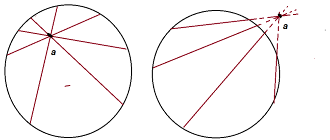

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- Subspaces of injectivity
- Inversion formula / procedure
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Eluded from attention:

Group action on B^n is behind the problem

Group-theoretical view; link with with the structure of hyperbolic space $\mathbb{H}^n$.

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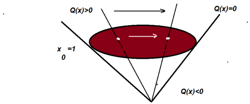
An universal approach: action of the hyperbolic group

Caley model of hyperbolic space

Lorentz group $S0(n,1)$: linear transf's of R^{n+1} preserving $Q(x) = x_0^2 - x_1^2 - ... - x_n^2$.

Identify $B^n = \{Q > 0\} \cap \{x_0 = 1\}$.



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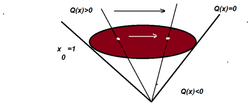

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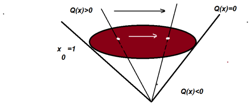

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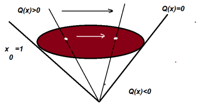


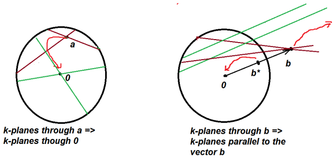
$SO(n,1)$ transitively acts on complexes of lines through 0 inside/outside the light cone, and therefore induces an automorphism group $Aut(B^n)$.

Important: $Aut(B^n)$ preserves affine sections of B^n ! Elements of $Aut(B^n)$ are fractional-linear mappings.

The strategy:

- **Interior center :** F_a, $|a| < 1$. Moving $a \to 0$ by $\varphi_a \in Aut(B^n)$, $\varphi_a(a) = 0$, delivers a link between F_a and F_0 (Central Funk transform).

- **Exterior center:** F_b, $|b| > 1$, Moving b to ∞, via sending the inverse point $b^* = \frac{b}{|b|^2} \to 0$ by $\varphi_{b^*} \in Aut(B^n)$, $\varphi_{b^*}(b^*) = 0$. Provides a bridge between F_b and Π_b (Parallel Slice Transform).



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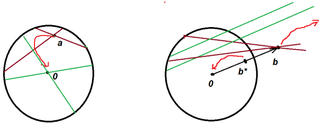

k-planes through a =>
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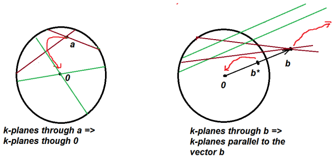

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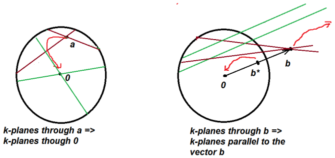

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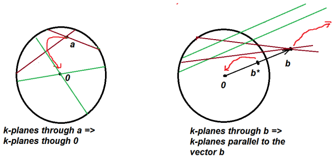

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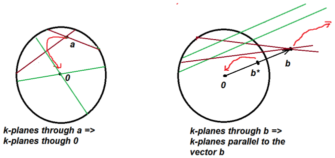

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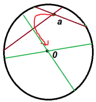

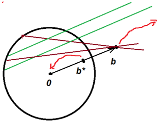

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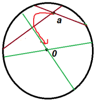

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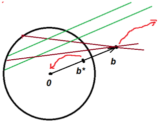

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Intertwining relations

Thm (B. Rubin, M.A; the case $|a| < 1, k = n - 1$- M. Quellmalz)

$$\boxed{F_a = \Phi_a F_0 M_a, \ |a| < 1.}$$

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where the intertwining operator is

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Kernels and inversion of the standard transforms

Well studied.

- $ker\ F_0 = \{$ odd functions $\}$.
- F_0^{-1} written.

- $ker \Pi_b = \{$ functions odd with respect to the hyperplane $\langle x, b \rangle = 0$.
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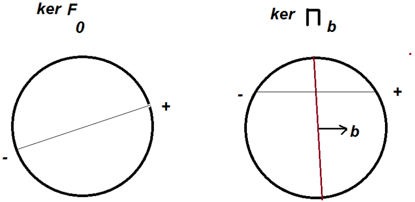

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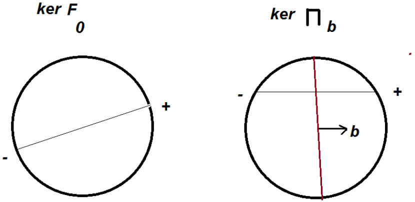

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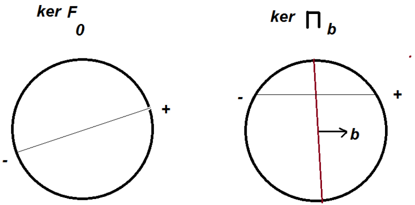

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Invoking known facts about F_0, Π_b, we obtain by means of the intertwining relations::
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- *Given* $a \in \mathbb{R}^n \setminus S^{n-1}$,

$$\text{ker } F_a = \{f \in C(S^{n-1}) : f(x) = -\rho_a(x) f(\tau_a x)\}$$

(a-odd functions).
*Here the a-***weight function** ρ_a *is*

$$\boxed{\rho_a(x) = \left(\frac{|1 - |a|^2|}{|x - a|^2}\right)^{k-1}}$$

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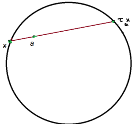

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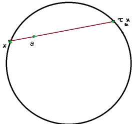

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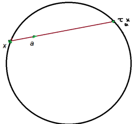

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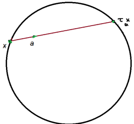

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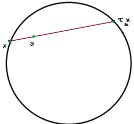

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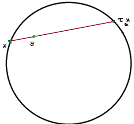

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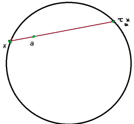

Inversion

- (Right) inverse operator for the **interior** center a:

$$\boxed{F_a^{-1} = M_a F_0^{-1} \Phi_a.}$$

It reconstructs the a-even part:

$$\boxed{F_a^{-1} F_a f = \frac{1}{2}\left(f + \rho_a(f \circ \tau_a)\right)}.$$

- (Right) inverse operator for the **exterior** center b:

$$\boxed{F_b^{-1} = M_b \Pi_b^{-1} \Phi_{b^*}},$$

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- the b-even part of $f \in C(S^{n-1})$.

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It reconstructs the a-even part:

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Multiple Funk transform

Given $A = \{a_1, ..., a_s\} \subset \mathbb{R}^n$ define

$$F_A f = (F_{a_1} f, ..., F_{a_s} f).$$

Q: *For what sets A the multiple Funk transform F_A is injective, i.e.,*

$$ker\ F_A = \cap_{j=1}^{s} ker\ F_{a_j} = \{0\}?$$

Motivation: Single Funk data $g_a = F_a$ is not enough to recover $f \in C(S^{n-1})$, unless $|a| = 1$. This gives rise to

Q: *What sets $g_a = F_a f,\ a \in A$, of Funk data uniquely determine f?*

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We will give answers for two-point sets $A = \{a, b\}$.

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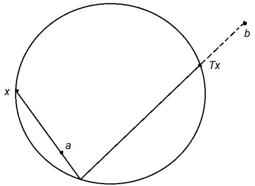
The double reflection billiard T

To formulate the injectivity result for the pairs, we need to define a V-mapping $T : S^{n-1} \to S^{n-1}$, by

$$\boxed{T = \tau_b \circ \tau_a :}$$



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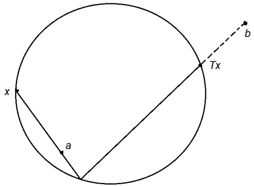

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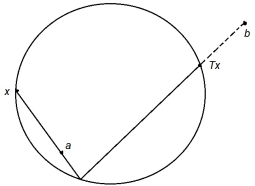

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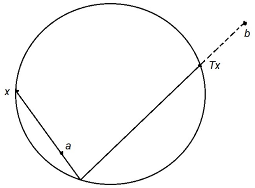

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In algebraic terms:

Thm *Given* $a,\ b \in \mathbb{R}^n$, *TFAE*

- *ker* $F_a \cap ker F_b = \{0\}$.
- *The group generated by the reflections* $\tau_a,\ \tau_b$ *is infinite.*
- *The mapping* T *is non-periodic, i.e.,* $\forall q \in \mathbb{N}$,

$$T^{\circ q} = \underbrace{T \circ \ldots \circ T}_{q} \neq \mathrm{id}.$$

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Analytic formulation

Thm *Given $a, b \in \mathbb{R}^n$, define*

$$\boxed{\Theta(a,b) = \frac{a \cdot b - 1}{\sqrt{(1-|a|^2)(1-|b|^2)}},}$$

$$\boxed{\kappa(a,b) := \frac{1}{\pi} \arccos \Theta(a,b).}$$

Then $\ker F_a \cap \ker F_b = \{0\}$ *if and only if one of the conditions is fulfilled:*

(i) $\Theta(a,b) \in \mathbb{C} \setminus [-1,1]$.

(ii) $\Theta(a,b) \in [-1,1]$ *but* $\kappa(a,b)$ *(rotation number) is irrational.*

In other words, the paired FT $F_{a,b} = (F_a,\ F_b)$ fails to be injective if and only if the rotation number $\kappa(a,b)$ is real rational.

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Reconstruction functions from a pair of Funk transforms

Given

$$F_a f = g_a, \; F_b f = g_b.$$

Then one can reconstruct:

$$F_a^{-1} g_a = \frac{1}{2}\left(f + \rho_a(f \circ \tau_a)\right)$$

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- *b*-even part. From here

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Substitute:

$$f = 2(F_b^{-1} g_b + W_b F_a^{-1} g_a) + W_b W_a f := h + Wf.$$

Iterate:

$$f = h + Wf = h + Wh + W^2 h = ... = \sum_{k=0}^{\infty} W^k h.$$

(formal series).

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$$f = h + Wf = h + Wh + W^2 h = \dots = \sum_{k=0}^{\infty} W^k h.$$

(formal series).

Reconstruction functions from a pair of Funk transforms

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- We know

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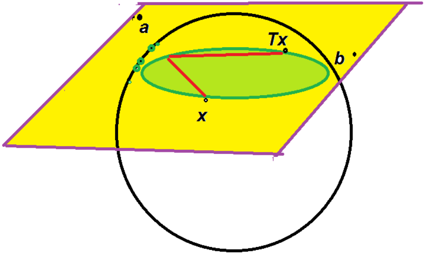
Reduction to T-dynamics on the unit circle

- The orbit $O_x = \{x, Tx, T^2x, ...\}$ entirely belongs to 2-dim plane $span(x,\ a,\ b)$.
- After a shift and re-scaling, the problem reduces to study of **complex** T-dynamics on the unit circle $S^1 \subset \mathbb{C}$.



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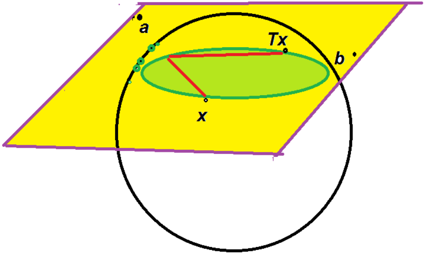

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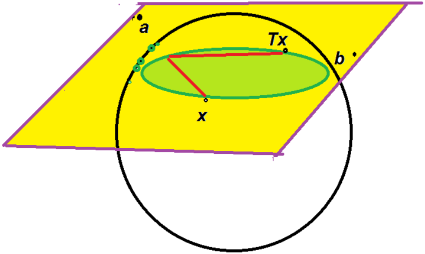

Complex dynamics on S^1

- T generates a complex Möbius transformation of S^1, associated with $T \in PSL(2, \mathbb{C})$.
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  In 1), 2), 3): two fixed points- attracting and repelling those. In elliptic case: T is conjugate with a rotation $z \to ze^{i\theta}$. Splits into: **rational** or **irrational** *rotation number* $\boxed{\frac{\theta}{2\pi} = \kappa(a, b).}$
- **Thm** The space of T-automorphic $C_T(S^1) = 0$ in cases 1), 2), 3). Elliptic case with irrational $\kappa(a, b)$- dense orbits.
- $C_T(S^1) \neq \{0\}$ for T for elliptic type with rational rotation number.
- Glueing up dynamics on 2D sections into global dynamics on S^{n-1}. (F_a, F_b) can be non-injective only for periodic T: elliptic case with rational $\kappa(a, b)$.
- For periodic $T : S^{n-1} \to S^{n-1}$ construct a non-zero $f \in ker(F_a, F_b)$.

Complex dynamics on S^1

- T generates a complex Möbius transformation of S^1, associated with $T \in PSL(2, \mathbb{C})$.
- Classification of the types of T-dynamics according to $\boxed{trace\ T = Theta(a, b).}$
  The cases: 1)*hyperbolic* , 2) *parabolic*, 3) *loxodromic*, 4) *elliptic*. Different types of orbits behaviour (convergence to attracting fixed points; dense orbits; finite orbits).
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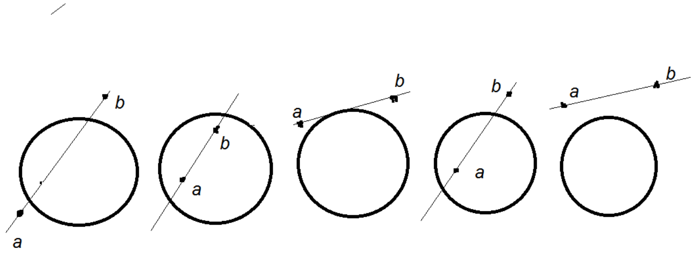
Illustration of the classification of the types of T - dynamics for $kerF_a \cap kerF_b = \{0\}$



hyperbolic type *parabolic type* *loxodromic type* *elliptic type*

<a, b > ≠ 1

irrational rotation number k(a,b)

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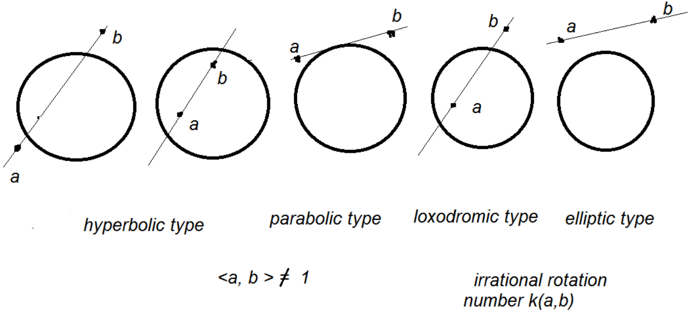

More than two centers

Q: Describe the common kernel of more than two shifted Funk transforms?

Q: *$ker\ F_A = ker\ F_{a_1} \cap ... \cap ker\ F_{a_s} =?,\ s > 2.$*

It follows that if $ker\ F_A \neq \{0\}$ then $G(A) := Group\left(\tau_{a_j}, j = 1, ..., s\right)$ is a Coxeter group $(\tau_{a_i}^2 = e,\ (\tau_{a_i}\tau_{a_j})^{q_{i,j}} = e.)$

Q: *Is the converse true?*

Does $G(A)$ being Coxeter group imply $ker\ F_A := \cap_{j=1}^{s} ker\ F_{a_j} \neq \{0\}$?

True: if $G(A)$ is finite then $ker\ F_A \neq \{0\}$.

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Thank you!