On a local solution to the 8th Busemann-Petty Problem

María Angeles Alfonseca
Joint work with F. Nazarov, D. Ryabogin and V. Yaskin

North Dakota State University

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Introduction

Let K be an origin symmetric convex body in $\mathbb{R}^n$.

Given $\theta \in S^{n-1}$, the unit sphere in $\mathbb{R}^n$, let $\theta^\perp$, be the hyperplane orthogonal to θ,

$$\theta^\perp = \{x \in \mathbb{R}^n : x \cdot \theta = 0\}.$$

For $\theta \in S^{n-1}$, we define the radial function of K,

$$\rho_K(\theta) = \sup\{t > 0 : t\theta \in K\}$$

and the support function of K,

$$h_K(\theta) = \sup\{\theta \cdot y : y \in K\}$$

We have $h_K = \frac{1}{\rho_{K^\circ}}$, where $K^\circ = \{x \in \mathbb{R}^n : x \cdot y \leq 1 \ \forall y \in K\}$ is the polar body of K.

5th Busemann-Petty Petty Problem

Assume that there exists a constant c_n such that for every $\theta \in S^{n-1}$,

$$h_K(\theta) vol_{n-1}(K \cap \theta^{\perp}) = c_n.$$

Does it follow that K is an ellipsoid?

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The answer is negative in dimension 2 (Radon): In this case,

$$vol_{n-1}(K \cap \theta^\perp) = 2\rho_K(\phi_{\pi/2}(\theta)),$$

and the equation becomes

$$\rho_{K^\circ}(\theta) = c\rho_K(\phi_{\pi/2}(\theta))$$

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Equation (1) is also invariant under linear transformations T (up to a factor of $|\det T|$), hence it is satisfied by ellipsoids.

Analytic Reformulation

The Intersection Body of K is defined by

$$\rho_{IK}(\theta) = vol_{n-1}(K \cap \theta^{\perp}),$$

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In polar coordinates,

$$\rho_{IK}(\theta) = \frac{1}{n-1} \int_{S^{n-1} \cap \theta^{\perp}} \rho_K^{n-1}(u) d\sigma(u) = c_n R(\rho_K^{n-1}),$$

where R is the spherical Radon transform, normalized so that $R(1) = 1$.

Thus, the equation

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The proof of the affirmative local result consists on the following steps:

1. The intersection body operator is a contraction in L^2 in a neighborhood of the Euclidean ball.
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(i) The intersection body operator is a contraction in L^2 in a neighborhood of the Euclidean ball. [Fish-Nazarov-Ryabogin-Zvavitch]

(ii) The polar intersection body operator defined by $K \to (IK)^{\circ}$ is also a contraction.

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(ii) The polar intersection body operator defined by $K \to (IK)^\circ$ is also a contraction.

This follows from (i) and a Maximal Function estimate for the polar body.

Maximal Function Estimate

Let M be the spherical Hardy-Littlewood maximal function,

$$Mf(\theta) = \sup_{\theta \in E} \frac{1}{\sigma(E)} \int_{S^{n-1} \cap E} |f(u)| d\sigma(u).$$

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Let $\rho_K = 1 + \chi$, with $\|\chi\|_2 < \epsilon$ and $\int_{S^{n-1}} \chi = 0$. We write χ in spherical harmonics,

$$\chi = \sum_{i=2}^{\ell} Y_i + \sum_{i=\ell+2}^{\infty} Y_i = \phi + \psi.$$

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Proposition:

Let K be close enough to the Euclidean ball in the Banach-Mazur distance. If $\rho_K = 1 + \phi + \psi$, then $h_K \approx 1 + \phi + M\psi$, where M is the spherical Hardy-Littlewood maximal function.

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Then,

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and by the maximal function estimate,

$$\leq\|1-\rho_{IK}\|_2+\|\rho_{IK}-h_{IK}\|_2\leq\|1-\rho_{IK}\|_2+c\|M\psi\|_2<\mu\epsilon,$$

where $\lambda<\mu<1$.

Iteration

Letting $K_2 := (IK)^\circ$ and $K_m := (IK_{m-1})^\circ$, we have

$$\|1 - \rho_{K_m}\|_2 \leq \mu \|1 - \rho_{K_{m-1}}\|_2,$$

where $0 < \mu < 1$.

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Since $(IK)^\circ = K$ by hypothesis,
we have $K_m = K$ for all m, which proves the result.

The 8th Busemann-Petty Problem

Busemann-Petty, 1956

"Are the ellipsoids characterized by the fact that the Gauss curvature at a point of contact with a tangent plane parallel to $\theta^\perp$ is proportional to $vol_{n-1}(K \cap \theta^\perp)^{-(n+1)}$?"

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$$f_K(\theta) = c_n vol_{n-1}(K \cap \theta^{\perp})^{n+1}. \tag{2}$$

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Analytic Reformulation of Busemann-Petty 8

If $h_K \in C^2(S^{n-1})$ and f_K is continuous and strictly positive, then

$$f_K = A(h_K),$$

where the operator A is defined as a sum of determinants of minors of the Hessian matrix of h_K.

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Thus, equation $f_K(\theta) = c_n vol_{n-1}(K \cap \theta^\perp)^{n+1}$ can be rewritten as

$$A(h_K) = c_n \left(R(\rho_K^{n-1}\right)^{n+1}.$$

A Local Solution to Busemann-Petty 8:

Assume that K is close enough to the Euclidean ball in the Banach-Mazur distance, and satisfies

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But for K close to the Euclidean ball,

$$h_K \approx \frac{1}{h_K},$$

and we have reduced Problem 8 to 5.

Linearizing the operator A

Lemma:

$$DA(1) = \Delta_{S^{n-1}} + (n-1)I,$$

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Spherical harmonics of degree m are eigenfunctions for $\Delta_{S^{n-1}}$, with eigenvalue $-m(m+n-2)$.

Lemma:

Let $\psi \in L^2(S^{n-1})$ be an even function such that $\int_{S^{n-1}} \psi = 0$. Then

$$(n+1)\|(\Delta_{S^{n-1}} + (n-1)I)^{-1}\psi\|_2 \leq \|\psi\|_2.$$

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Proof: Let

$$\psi = \sum_{m\geq 2, even}^{\infty} a_m Ym$$

be the decomposition of ψ in spherical harmonics. By Parseval,

$$\|(\Delta_{S^{n-1}} + (n-1)I)^{-1}\psi\|_2 = \left(\sum_{m\geq 2, even}^{\infty} \frac{a_m^2}{(-m(m+n-2)+n-1)^2} \right)^{1/2}$$

$$\leq \left(\sum_{m\geq 2, even}^{\infty} \frac{a_m^2}{(n+1)^2} \right)^{1/2} = \frac{1}{n+1}\|\psi\|_2.$$

To finish the proof, it remains to estimate

$$\|A - DA(1)\|_{L^2(S^{n-1})},$$

which is done using the theory of singular integrals.

Thank you!