Radon transforms supported in hypersurfaces and a conjecture by Arnold

Jan Boman

BIRS, Febr 10, 2020

Plan of talk

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Sketch of proof of the theorem

The plane Radon Transform

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$$Rf(\omega, p) = \int_{x\cdot\omega=p} f\,ds, \quad (\omega, p) \in S^1 \times \mathbb{R},$$

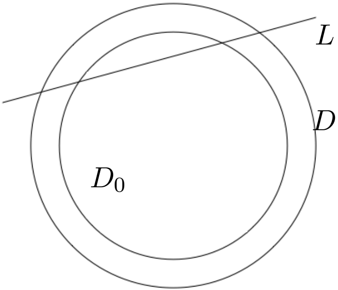
where the line L is defined by $x \cdot \omega = p$. Clearly

$$Rf(\omega, p) = Rf(-\omega, -p).$$

The Interior Radon Transform

Given two concentric disks D and $\overline{D_0} \subset D$ it is well known that there exists a non-trivial function f with support *equal* to $\overline{D}$ such that

$$Rf(L) = 0 \quad \text{for all lines } L \text{ that meet } D_0.$$



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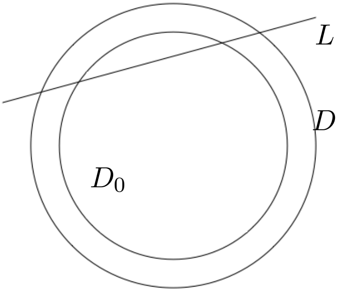

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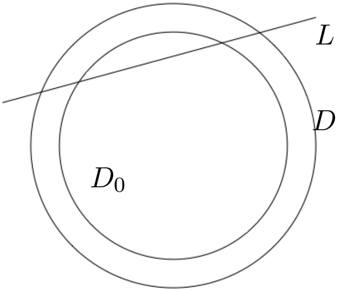

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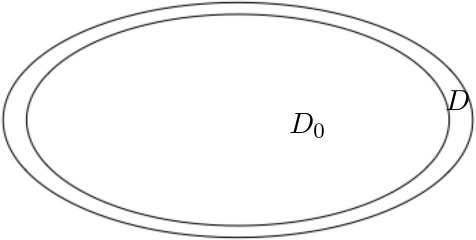
One can prescribe $g(p)$ arbitrarily and find $f(r)$ so that $Rf(p) = g(p)$, for instance choose $g(p) = 0$ for $|p| \leq p_0 < 1$.

More precisely

$$f(r) = \frac{-1}{\pi} \int_r^1 (s^2 - r^2)^{-1/2} g'(s) ds.$$

The Interior Radon Transform, cont.

Same for ellipses:



There exist functions f supported in D such that $Rf(L) = 0$ for all lines L that intersect D_0.

The Interior Radon Transform, cont.

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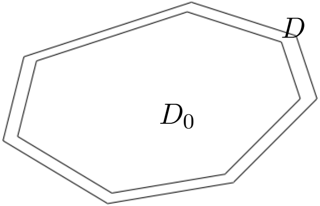
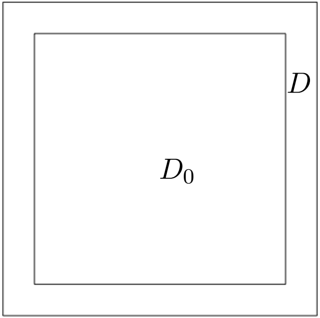
Conjecture. Let D and D_0 be bounded convex domains in the plane with $\overline{D_0} \subset D$. Then there exists a smooth function f, not identically zero, $\operatorname{supp} f \subset \overline{D}$, such that its Radon transform $Rf(L)$ vanishes for every line L that intersects D_0.



For instance squares:



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This leads to the more general question, closely related to the Conjecture:

How can the support of a Radon transform look?

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This leads to the more general question, closely related to the Conjecture:

How can the support of a Radon transform look?

Or more precisely:

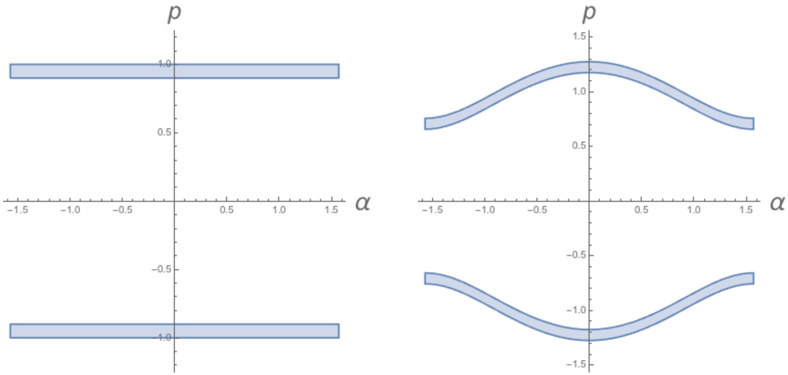
Which subsets of the manifold of lines in the plane can be the support of Rf for some compactly supported function or distribution f in $\mathbb{R}^2$?

Identify the set of lines in the plane with the set of pairs

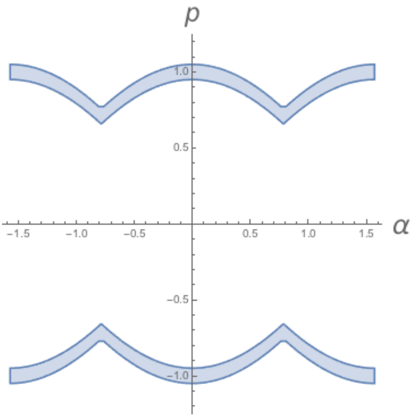
$$(\omega, p) \in S^1 \times \mathbb{R}, \quad \text{where} \quad (\omega, p) \sim (-\omega, -p)$$

and write $\omega = (\cos\alpha, \sin\alpha)$.

We saw that the support of a Radon transform can look as below.
Left: concentric disks. Right: concentric ellipses.



If the conjecture is true, the support of a Radon transform can look like this:



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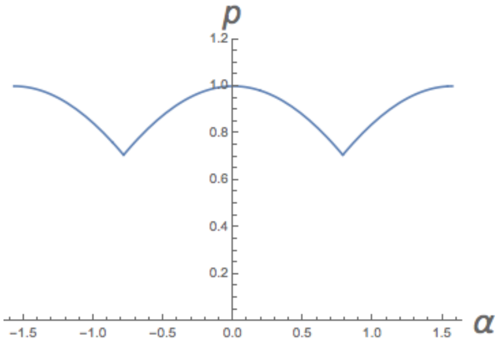
The curve we saw in the previous picture is the graph of the supporting function for a centered square:



A Radon transform supported on a curve (in the mfd of lines)

Let f_0 be the function in the plane defined by

$$f_0(x) = \frac{1}{\pi} \frac{1}{\sqrt{1 - |x|^2}} \quad \text{for } |x| < 1$$

and $f = 0$ for all other $x = (x_1, x_2)$.

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$$Rf_0(\omega, p) = \int_{x \cdot \omega = p} f_0(x)\, ds = 1 \quad \text{for } |p| < 1,$$

and obviously $Rf_0(\omega, p) = 0$ for $|p| \geq 1$.

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Let f be the distribution $f = \Delta f_0 = (\partial_{x_1}^2 + \partial_{x_2}^2) f_0$.

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Now use the well known formula $R(\Delta h)(\omega, p) = \partial_p^2 Rh(\omega, p)$ with $h = f_0$.

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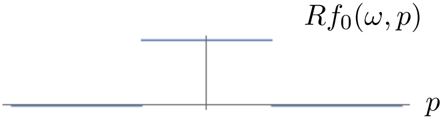

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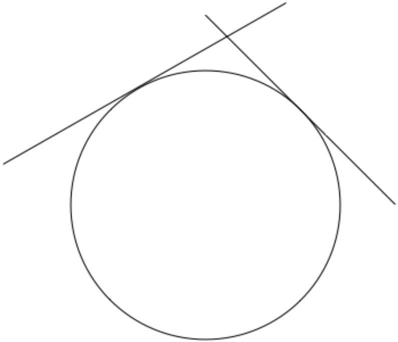
This means that the distribution $f = \Delta f_0$ has the property that its Radon transform, a distribution on the manifold of lines in the plane, must be supported on the set of tangents to the unit circle.



The Radon transform of a *distribution* f in $\mathbb{R}^n$ is defined by

$$\langle Rf, \varphi \rangle = \langle f, R^* \varphi \rangle, \quad \text{for all test functions } \varphi \text{, where}$$

$$(R^*\varphi)(x) = \int_{S^{n-1}} \varphi(\omega, x \cdot \omega) d\omega,$$

$d\omega$ is surface measure on S^{n-1}, or

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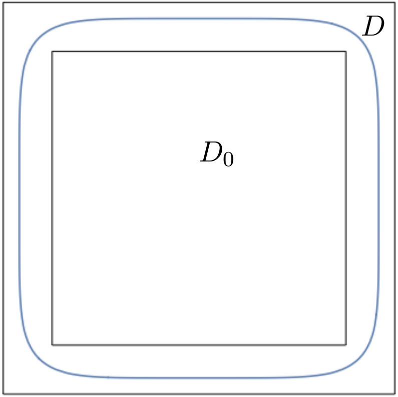
By means of an affine transformation we can easily construct a similar example where D is an ellipse.

Proof idea for Conjecture (here shown for case of squares): find a compactly supported distribution f whose Radon transform is supported on the set of tangents to the blue curve.



However: to my surprise I found the following:

Theorem 1 (JB 2018). Let $D \subset \mathbb{R}^n$ be a bounded, convex domain. Assume that there exists a distribution $f \neq 0$, supported in $\overline{D}$, such that Rf is supported in the set of supporting planes to ∂D. Then the boundary of D is an ellipsoid.

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If ∂D is C^1 smooth, the supporting planes for D are of course tangent planes to ∂D.

Newton's lemma

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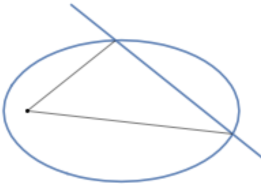
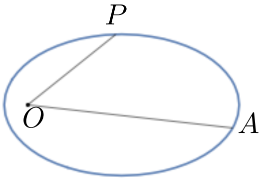
Lemma 28 in Newton’s *Principia* reads according to Arnold and Vassiliev in *Newton’s Principia read 300 years later* (Notices of the AMS 1989):

Lemma. There exists no algebraically integrable convex non-singular algebraic curve.

Newton's lemma, cont.





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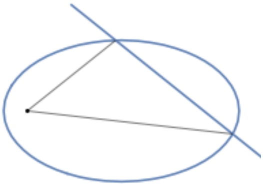
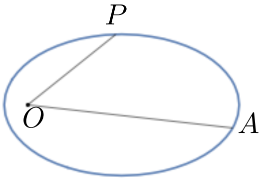
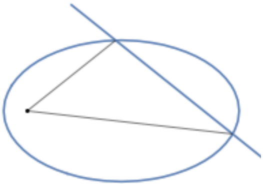
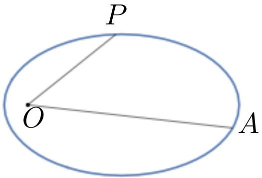




A segment is equal to a sector minus a triangle, and the area of the triangle depends algebraically on the coordinates of the corners.

Newton’s lemma, cont.





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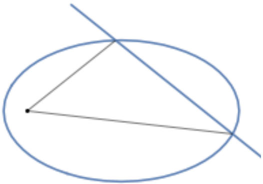
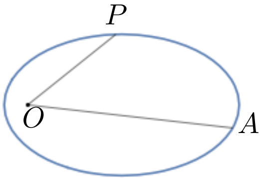
Let A be fixed, and let $f(P)$ be the area of the sector defined by the lines OA and OP.

Newton’s lemma, cont.





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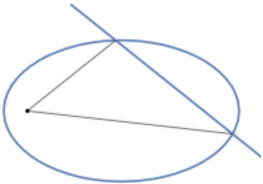

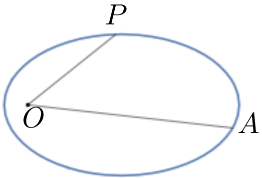

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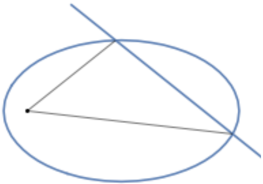
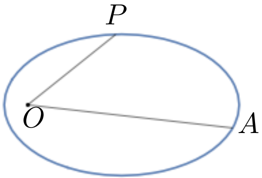
Newton's lemma, cont.





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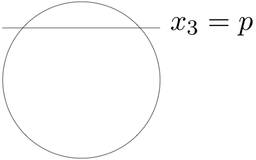
So the function $f(P)$ must have infinitely many values, which is impossible if it is algebraic.

Higher dimensions: the case of odd dimension



The volume of the part of the unit ball in $\mathbb{R}^3$ that lies above the plane $x_3 = p$ is

$$\int_p^1 \pi(\sqrt{1-t^2})^2 dt = \int_p^1 \pi(1-t^2) dt = \frac{\pi}{3}(p^3 - 3p + 2).$$

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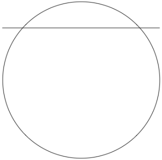

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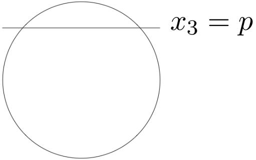

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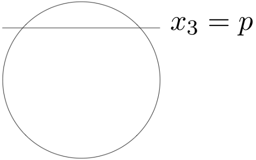

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And same for ellipsoids.

Arnold's Conjecture

Problem 1987-14 in *Arnold's Problems* reads:

Do there exist smooth hypersurfaces in $\mathbb{R}^n$ (other than the quadrics in odd-dimensional spaces), for which the volume of the segment cut by any hyperplane from the body bounded by them is an algebraic function of the hyperplane?

The case of even dimension

Theorem 2 (Vassiliev 1988). There exist no smooth, convex algebraically integrable bounded domains in even dimensions.

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V. A. Vassiliev: *Applied Picard - Lefschetz Theory*, AMS 2002.

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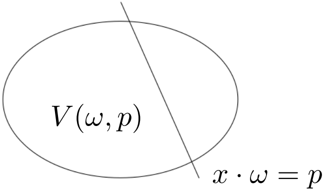
Denote by $V(\omega, p)$ the volume cut out from the domain D by the hyperplane $x \cdot \omega = p$. Assume that $p \mapsto V(\omega, p)$ is a *polynomial* for every ω. Prove that the boundary of D must be an ellipsoid.



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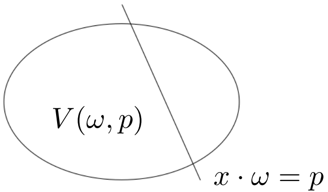

Theorem 3 (Koldobsky, Merkurjev, and Yaskin 2017). Assume that D is convex and has C^∞ boundary and that $p \mapsto V(\omega, p)$ is a polynomial of degree $\leq N$ for every ω. Then the boundary of D must be an ellipsoid.

Recall:

Theorem 1 (JB 2018). Let $D \subset \mathbb{R}^n$ be a bounded, convex domain. Assume that there exists a distribution $f \neq 0$, supported in $\overline{D}$, such that Rf is supported in the set of supporting planes to ∂D. Then the boundary of D is an ellipsoid.

Theorem 1 implies Theorem 3

Let $\chi_D(x)$ be the characteristic function for the domain D and let $V(\omega, p)$ be the volume function discussed earlier.

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$$R(\Delta^k \chi_D)(\omega, p) = \partial_p^{2k} R\chi_D(\omega, p).$$

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Since $p \mapsto V(\omega, p)$ is polynomial (for p such that the plane $x \cdot \omega = p$ intersects D) and $\partial_p V(\omega, p) = (R\chi_D)(\omega, p)$, it follows that $p \mapsto R(\chi_D)(\omega, p)$ is polynomial, so $\partial_p^{2k} R\chi_D(\omega, p) = 0$ if k is large enough except at the jump points, which correspond to tangent planes.

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Remark 1. Theorem 1 implies Theorem 3 without the smoothness assumption on the boundary of D.

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Remark 2. Theorem 3 shows that the Radon transform of the characteristic function χ_D cannot be polynomial unless ∂D is an ellipsoid. Theorem 1 shows that *no function* supported in D can have a polynomial Radon transform unless ∂D is an ellipsoid.

The range of the Radon transform

Which functions $g(\omega, p)$ on $S^1 \times \mathbb{R}$ are equal to $Rf(\omega, p)$ for some compactly supported function f?

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Which functions $g(\omega, p)$ on $S^1 \times \mathbb{R}$ are equal to $Rf(\omega, p)$ for some compactly supported function f?

An even function $g(\omega, p)$ is equal to $Rf(\omega, p)$ for some compactly supported function f if and only if the function

$$S^1 \ni \omega \mapsto \int_{\mathbb{R}} g(\omega, p) p^k dp$$

is equal to the restriction to S^1 of a homogeneous polynomial in (ω_1, ω_2) of degree k for every natural number k.

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$$\int_{\mathbb{R}} g(p)p^k dp = 0 \quad \text{for all odd } k, \quad \text{and}$$
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And since $1 = \omega_1^2 + \omega_2^2$ for $\omega \in S^1$, the constant function is the restriction to S^1 of a homogeneous polynomial of an arbitrary even degree.

Distributions supported on the set of supporting planes

Assume for simplicity that $D = -D$. Let $\rho(\omega)$ be the supporting function for D

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If $q(\omega)$ is an even function on S^{n-1}, then

$$g(\omega, p) = q(\omega)\big(\delta(p - \rho(\omega)) + \delta(p + \rho(\omega))\big)$$

defines a distribution (of order zero) on the manifold of hyperplanes.

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defines a distribution (of order zero) on the manifold of hyperplanes. More generally, if $g = Rf$, f compactly supported, and g is supported on $p = \pm\rho(\omega)$, then $g(\omega, p)$ can be written

$$g(\omega, p) = \sum_{j=0}^{m-1} q_j(\omega)\big(\delta^{(j)}(p - \rho(\omega)) + (-1)^j \delta^{(j)}(p + \rho(\omega))\big),$$

for some even distributions q_j, $q_j(\omega) = q_j(-\omega)$, on the sphere S^{n-1}.

Plan of proof of Theorem 1

1. Write down the condition that $\int_{\mathbb{R}} g(\omega, p)p^k dp$ is a polynomial of degree k in ω for each k.

2. Prove that those conditions imply that $\rho(\omega)^2$ must be a quadratic polynomial.

To compute

$$\int_{\mathbb{R}} g(\omega, p) p^k dp$$

we use for instance the fact that

$$\begin{aligned}\int_{\mathbb{R}} \delta'(p - \rho(\omega)) p^k dp &= -\int_{\mathbb{R}} \delta(p - \rho(\omega))\, k\, p^{k-1} dp \\ &= -k\, \rho(\omega)^{k-1}.\end{aligned}$$

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Recall that

$$g(\omega, p) = \sum_{j=0}^{m-1} q_j(\omega)\big(\delta^{(j)}(p-\rho(\omega)) + (-1)^j \delta^{(j)}(p+\rho(\omega))\big).$$

The range conditions therefore mean that there must exist polynomials p_0, p_2, p_4 etc., where $p_k(\omega)$ is homogeneous of degree k, such that (for instance if $m = 3$)

$$
\begin{aligned}
&q_0 = p_0\\
&q_0\rho^2 + 2\,q_1\rho + 2\,q_2 = p_2\\
&q_0\rho^4 + 4\,q_1\rho^3 + 4\cdot 3\,q_2\rho^2 = p_4\\
&q_0\rho^6 + 6\,q_1\rho^5 + 6\cdot 5\,q_2\rho^4 = p_6\\
&q_0\rho^8 + 8\,q_1\rho^7 + 8\cdot 7\,q_2\rho^6 = p_8\\
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\end{aligned}
$$

Let us write this in matrix form.

$$
\begin{pmatrix}
1 & 0 & 0 \\
\rho^2 & 2\rho & 2 \\
\rho^4 & 4\rho^3 & 4\cdot 3\rho^2 \\
\rho^6 & 6\rho^5 & 6\cdot 5\rho^4 \\
\rho^8 & 7\rho^7 & 8\cdot 7\rho^6 \\
\rho^{10} & 10\rho^9 & 10\cdot 9\rho^8 \\
\ldots & \ldots & \ldots
\end{pmatrix}
\begin{pmatrix} q_0 \\ q_1 \\ q_2 \end{pmatrix}
=
\begin{pmatrix} p_0 \\ p_2 \\ p_4 \\ p_6 \\ p_8 \\ \ldots \end{pmatrix}.
$$

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Forming suitable linear combinations of four of those equations we can eliminate the q-functions. This gives infinitely many equations of the form

$$
\begin{aligned}
\rho^6 p_0 - 3\rho^4 p_2 + 3\rho^2 p_4 &= p_6 \\
\rho^6 p_2 - 3\rho^4 p_4 + 3\rho^2 p_6 &= p_8 \\
\rho^6 p_4 - 3\rho^4 p_6 + 3\rho^2 p_8 &= p_{10} \\
\rho^6 p_6 - 3\rho^4 p_8 + 3\rho^2 p_{10} &= p_{12} \\
\dots
\end{aligned}
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\dots
\end{aligned}
$$

We now have only two kinds of functions of ω: the supporting function $\rho(\omega)$ and the polynomials $p_k(\omega)$. The only known fact is that $p_k(\omega)$ is a homogeneous polynomial in ω of degree k for every k.

Considering the first three equations as a linear system in the three "unknowns" ρ^2, ρ^4, and ρ^6, we can write those equations

$$\begin{pmatrix} p_0 & p_2 & p_4 \\ p_2 & p_4 & p_6 \\ p_4 & p_6 & p_8 \end{pmatrix} \begin{pmatrix} \rho^6 \\ -3\rho^4 \\ 3\rho^2 \end{pmatrix} = \begin{pmatrix} p_6 \\ p_8 \\ p_{10} \end{pmatrix}. \tag{1}$$

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Provided the determinant of the matrix is different from zero, we can solve for instance ρ^2 from this system and obtain ρ^2 as a rational function

$$\rho(\omega)^2 = \frac{F(\omega)}{G(\omega)},$$

where $F(\omega)$ and $G(\omega)$ are polynomials, and

$$G(\omega) = \det \begin{pmatrix} p_0 & p_2 & p_4 \\ p_2 & p_4 & p_6 \\ p_4 & p_6 & p_8 \end{pmatrix}.$$

Considering the first three equations as a linear system in the three "unknowns" ρ^2, ρ^4, and ρ^6, we can write those equations

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However, with very little additional effort we can do much better.

The following identities are trivial.

$$\begin{pmatrix} p_0 & p_2 & p_4 \\ p_2 & p_4 & p_6 \\ p_4 & p_6 & p_8 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} p_2 \\ p_4 \\ p_6 \end{pmatrix} \quad \text{and}$$

$$\begin{pmatrix} p_0 & p_2 & p_4 \\ p_2 & p_4 & p_6 \\ p_4 & p_6 & p_8 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} p_4 \\ p_6 \\ p_8 \end{pmatrix}.$$

Combining the linear system (1) with those two trivial equations we obtain the matrix equation

$$\begin{pmatrix} p_0 & p_2 & p_4 \\ p_2 & p_4 & p_6 \\ p_4 & p_6 & p_8 \end{pmatrix} \begin{pmatrix} 0 & 0 & \rho^6 \\ 1 & 0 & -3\rho^4 \\ 0 & 1 & 3\rho^2 \end{pmatrix} = \begin{pmatrix} p_2 & p_4 & p_6 \\ p_4 & p_6 & p_8 \\ p_6 & p_8 & p_{10} \end{pmatrix}.$$

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The advantage with this equation is that it can be iterated. Setting

$$A = \begin{pmatrix} 0 & 0 & \rho^6 \\ 1 & 0 & -3\rho^4 \\ 0 & 1 & 3\rho^2 \end{pmatrix}$$

we have

$$\begin{pmatrix} p_0 & p_2 & p_4 \\ p_2 & p_4 & p_6 \\ p_4 & p_6 & p_8 \end{pmatrix} A^2 = \begin{pmatrix} p_4 & p_6 & p_8 \\ p_6 & p_8 & p_{10} \\ p_8 & p_{10} & p_{12} \end{pmatrix}.$$

And more generally

$$\begin{pmatrix} p_0 & p_2 & p_4 \\ p_2 & p_4 & p_6 \\ p_4 & p_6 & p_8 \end{pmatrix} A^k = \begin{pmatrix} p_{2k} & p_{2k+2} & p_{2k+4} \\ p_{2k+2} & p_{2k+4} & p_{2k+6} \\ p_{2k+4} & p_{2k+6} & p_{2k+8} \end{pmatrix}$$

for every k.

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for every k. The determinant of A is $\rho(\omega)^6$. It follows that

$$G(\omega)\rho(\omega)^{6k} \quad \text{is a polynomial for every } k.$$

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for every k. The determinant of A is $\rho(\omega)^6$. It follows that

$$G(\omega)\rho(\omega)^{6k} \quad \text{is a polynomial for every } k.$$

Since we already knew that $\rho(\omega)^2$ is a rational function, we can now conclude that $\rho(\omega)^2$ must be a polynomial (still assuming that $G(\omega)$ is not identically zero).

Therefore it remains only to prove

Lemma. If $q_{m-1} \neq 0$, then the $m \times m$ matrix

$$\begin{pmatrix} p_0 & p_2 & p_4 & \cdots & p_{m-2} \\ p_2 & p_4 & p_6 & \cdots & p_m \\ p_4 & p_6 & p_8 & \cdots & p_{m+2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{m-2} & p_m & p_{m+2} & \cdots & p_{2m-4} \end{pmatrix}$$

is non-singular.

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is non-singular.

This fact depends on the spectral properties of the matrix A.

A semi-local result

Theorem 4. Let D be open, convex, bounded, let $x^0 \in \partial D$, and let ω^0 be one of the unit normals of a supporting plane L_0 to $\overline{D}$ at x^0. If there exists a distribution f with support in $\overline{D}$ and a translation invariant open neighborhood W of L_0, such that the restriction of the distribution Rf to W is supported on the set of supporting planes to D in W, then ∂D must be equal to the restriction of an ellipsoid in some neighborhood of $\pm x^0$.

A recent, somewhat related, result:

Theorem (Ilmavirta and Paternain, 2018). Let $D \subset \mathbb{R}^n$ be a bounded and strictly convex domain with smooth boundary. If there exists a function $f \in L^1(D)$ such that the integral of f over almost every line meeting D is equal to 1, then D is a ball.

Agranovsky, M. *On polynomially integrable domains in Euclidean spaces*, in *Complex analysis and dynamical systems*, 1-21, Trends Math., Birkhäuser/Springer 2018.

Agranovsky, M. *On algebraically integrable bodies*, in *Functional Analysis and Geometry,* Selim Krein Centennial, Amer. Math. Soc., Providence, RI 2019.

Arnold, V. and Vassilev, V. A., *Newton's Principia read 300 years later*, Notices Amer. Math. Soc. **36** (1989), 1148-1154.

Vassiliev, V. A., *Applied Picard-Lefschetz theory*, Amer. Math. Soc. 2002.

Boman, J., A hypersurface containing the support of a Radon transform transform must be an ellipsoid. I, to appear in J. Geom. Anal.

Koldobsky, A., Merkurjev, A., and Yaskin, V., *On polynomially integrable convex bodies*, Adv. Math. **320** (2017), 876-886.

Ilmavirta, J. and Paternain, G., *Functions of constant geodesic X-ray transform,* Inverse Problems **35** (2019), 065002.