Volume product and metric spaces

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Joint work with Mattew Alexander, Matthieu Fradelizi and Artem Zvavitch

Kent State University

Workshop in Geometric Tomography
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Región de Murcia

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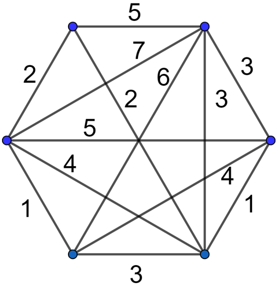

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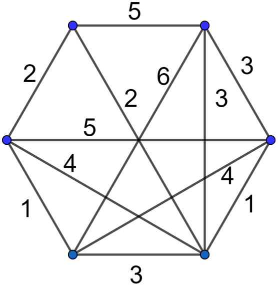

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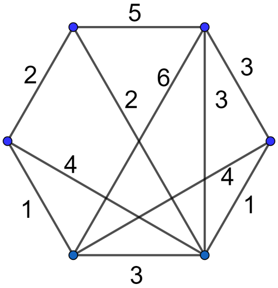

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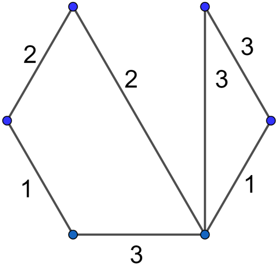

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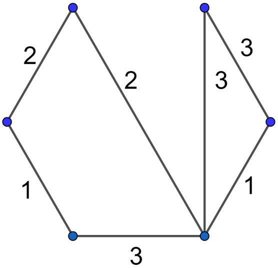

- (x, y) is an edge of the graph if and only if $d(x, y) < d(x, z) + d(z, y)$ for all $z \in M \setminus \{x, y\}$.

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$$B_{\mathrm{Lip}_0(M)} := \left\{ f : \frac{f(a_i) - f(a_j)}{d(a_i, a_j)} \leq 1 \ \forall i \neq j \right\} = \left\{ f : \langle f, \frac{e_i - e_j}{d(a_i, a_j)} \rangle \leq 1 \ \forall i \neq j \right\}$$

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- $\mathcal{F}(M)$ is called the *Lipschitz-free space over M* (also Arens-Eells, Wasserstein 1, transportation cost, Kantorovich-Rubinstein, ...)

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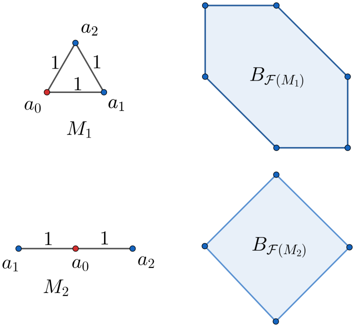

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Theorem (Aliaga–Guirao, 2019)

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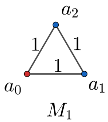

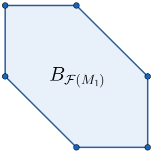

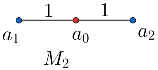

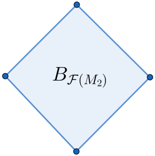

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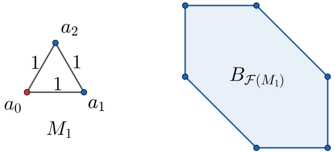

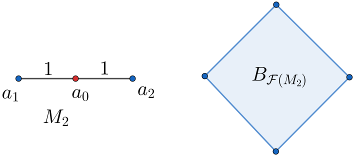

Indeed, we have shown that $\frac{e_i - e_j}{d(a_i,a_j)}$ belongs to a face of $B_{\mathcal{F}(M)}$ of dimension k precisely if there are k different points $z_1, \ldots, z_k \in M \setminus \{x,y\}$ such that $d(x,y) = d(x,z_k) + d(z_k,y)$.

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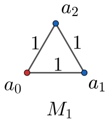

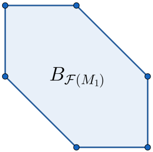

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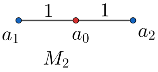

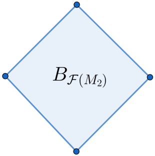

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Theorem (Godard, 2010)

- *M is a tree if and only if $B_{\mathcal{F}(M)}$ is a linear image of B_1^n.*
- *M embeds into a tree if and only if $B_{\mathrm{Lip}_0(M)}$ is a zonoid.*

Volume product

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Blaschke–Santaló inequality

$$\mathcal{P}(K) \leq \mathcal{P}(B_2^n)$$

- (Blaschke, 1923) for $n \leq 3$, (Santaló, 1948) for $n > 3$.
- (Saint-Raymond, 1981), (Petty, 1985) for the equality case.
- Proofs using Steiner symmetrization: (Ball, 1986), (Meyer–Pajor, 1990).
- Harmonic Analysis based proof (Bianchi–Kelly, 2015).
- Stability Results: (Böröczky, 2010), (Barthe–Böröczky–Fradelizi, 2014).
- Functional forms (for log-concave functions): (Ball, 1986), (Artstein-Avidan –Klartag–Milman, 2004), (Fradelizi–Meyer, 2007), (Lehec, 2009).

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- Around Hanner polytopes/Unconditional bodies (Nazarov–Petrov–Ryabogin–Zvavitch, 2010), (Kim, 2013), (Kim–Zvavitch, 2013).
- Body has a point of positive curvature then it is not a minimizer. (Stancu, 2009), (Reisner–Schütt–Werner, 2010), (Gordon–Meyer, 2011).
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- Hyperplane sections of ℓ_p-balls and Hanner polytopes, (Karasev, 2019).
- Convex bodies with 'many' symmetries (Barthe–Fradelizi, 2010).
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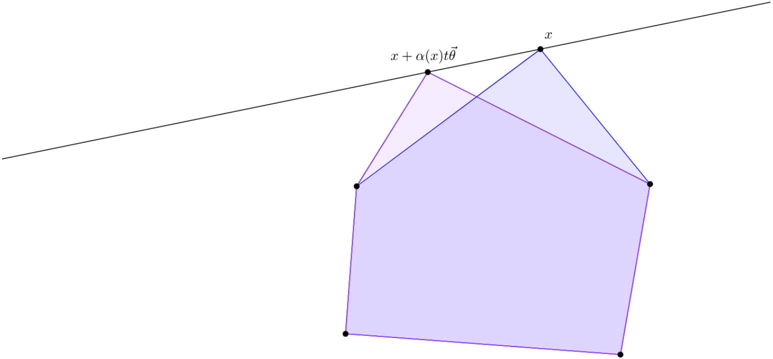

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As a consequence, if $t \mapsto |K_t|$ is affine, then

$$\min_{t \in [a,b]} \mathcal{P}(K_t) = \min\{\mathcal{P}(K_a), \mathcal{P}(K_b)\}$$

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Let M be a finite metric space with minimal volume product such that $B_{\mathcal{F}(M)}$ is a simplicial polytope. Then M is a tree (and so $\mathcal{P}(M) = 4^n/n!$).

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$$B_{\mathcal{F}(M_t)} = \mathrm{conv}\left\{\left(\mathrm{vertices}(B_{\mathcal{F}(M)}) \setminus \{\pm m_{ij}\}\right) \cup \pm(1+t)m_{ij}\right\}$$

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Thus, $B_{\mathcal{F}(M)}$ is a double cone with respect to each one of its vertices. So it is a linear image of B_1^n.

Hanner polytopes

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Consider convex symmetric bodies $K \subset \mathbb{R}^{n_1}$ and $L \subset \mathbb{R}^{n_2}$ denote by:

- $K \oplus_\infty L = K + L$ their ℓ_∞-sum: $\|(x_1, x_2)\|_{K \oplus_\infty L} = \max\{\|x_1\|_K, \|x_2\|_L\}$
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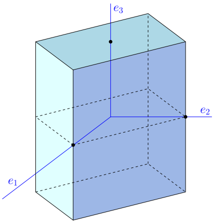
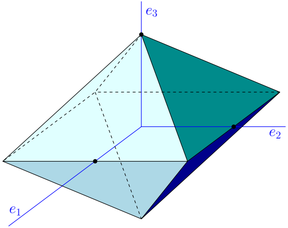


$(I \oplus_1 I) \oplus_\infty I$



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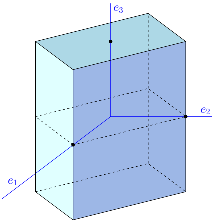

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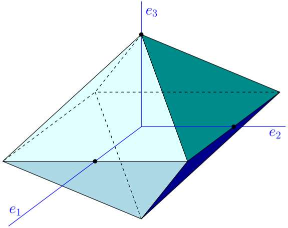

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A symmetric convex body is called a **Hanner polytope** if it is one-dimensional, or the ℓ_1 or ℓ_∞ sum of two (lower dimensional) Hanner polytopes.

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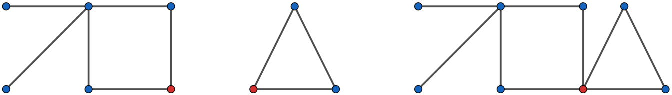

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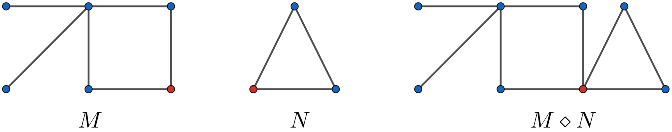

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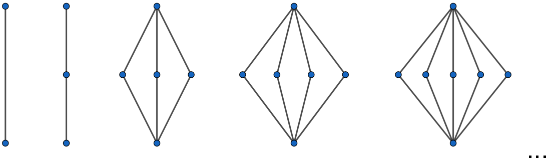
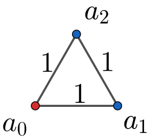
Theorem (Alexander–Fradelizi–G.–Zvavitch, 2019)

$B_{\mathcal{F}(M)}$ is a Hanner polytope if and only if $M = M_1 \diamond \ldots \diamond M_r$ and each M_i either contains only two points or it is the complete bipartite graph $K_{2,n}$, where all the edges have the same weight.



Maximal volume product

For $n = 2$, the metric space with maximum volume product is a complete graph with equal weights.



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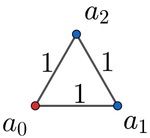

Theorem (Alexander–Fradelizi–G.–Zvavitch, 2019)

Assume that $\mathcal{P}(M)$ is maximal among the metric spaces with the same number of elements. Then

- $d(x, y) < d(x, z) + d(z, y)$ *for all different points* $x, y, z \in M$, *and*
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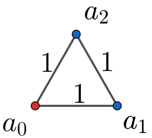

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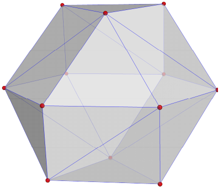
If $n \geq 3$ and M is the complete graph with equal weights, then $B_{\mathcal{F}(M)}$ is not simplicial!



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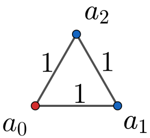

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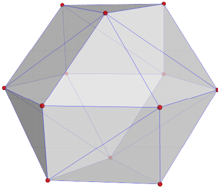

Thank you for your attention