Slicing properties of the lattice point enumerator

Martin Henk

based on ongoing joint work with Ansgar Freyer

BIRS, February 2020

- Loomis, Whitney, 1949. Let $K \subset \mathbb{R}^n$ be measurable. Then

$$\mathrm{vol}\,(K)^{\frac{n-1}{n}} \leqslant \left(\prod_{i=1}^{n} \mathrm{vol}_{\,n-1}(K|\boldsymbol{e}_i^{\perp}) \right)^{\frac{1}{n}}.$$

Equality holds e.g. for the box $K = [a_1, b_1] \times \cdots \times [a_n, b_n]$.

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  Equality holds e.g. for the box $K = [a_1, b_1] \times \cdots \times [a_n, b_n]$.
- Or, equivalently

$$\mathrm{G}(K)^{\frac{n-1}{n}} \leqslant \left(\prod_{i=1}^{n} \mathrm{G}(K|\boldsymbol{e}_i^{\perp}) \right)^{\frac{1}{n}},$$

  where $\mathrm{G}(K) = \#(K \cap \mathbb{Z}^n)$ denotes the lattice point enumerator (Schwenk, Munro, 1983).

- Meyer, 1988. Let $K \subset \mathbb{R}^n$ be a convex body. Then

$$\mathrm{vol}\,(K)^{\frac{n-1}{n}} \geqslant \frac{n!^{\frac{1}{n}}}{n} \left(\prod_{i=1}^{n} \mathrm{vol}_{\,n-1}(K \cap \boldsymbol{e}_i^{\perp}) \right)^{\frac{1}{n}},$$

and equality holds iff $K = \mathrm{conv}\,\{-a_i\boldsymbol{e}_i, b_i\boldsymbol{e}_i : 1 \leqslant i \leqslant n\}$.

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- Gardner, Gronchi, Zong, 2005.
  - Does there exist a discrete analog of Meyer's inequality?
  - For $n = 2$ it holds

$$\mathrm{G}(K)^{\frac{1}{2}} > \frac{1}{\sqrt{3}}\left(\prod_{i=1}^{2}\mathrm{G}(K \cap \boldsymbol{e}_i^{\perp})\right)^{\frac{1}{2}},$$

and it is best possible: $\mathrm{conv}\,\{\pm\boldsymbol{e}_1, \pm\, m\boldsymbol{e}_2\}$, $m \in \mathbb{N}$.

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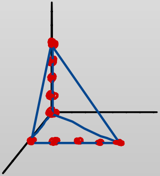
$$\frac{\mathrm{G}(S)^{\frac{n-1}{n}}}{\prod_{i=1}^{n} \mathrm{G}(S \cap \boldsymbol{e}_i^{\perp})^{\frac{1}{n}}} \leqslant 2(m+1)^{-1/n} \to_m 0.$$



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- So we have to assume $K = -K$.

- For the class of o-symmetric convex bodies $\mathcal{K}_o^n$ let

$$c(n) = \inf\left\{ \frac{\mathrm{G}(K)^{\frac{n-1}{n}}}{\prod_{i=1}^{n} \mathrm{G}(K \cap \boldsymbol{e}_i^{\perp})^{\frac{1}{n}}} : K \in \mathcal{K}_o^n \right\}.$$

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  - $$\frac{\mathrm{G}(K)^{\frac{n-1}{n}}}{\prod_{i=1}^n \mathrm{G}(K \cap \boldsymbol{e}_i^\perp)^{\frac{1}{n}}} = \frac{(3^{n-1} + 2m)^{\frac{n-1}{n}}}{3^{\frac{n-1}{n}}(3^{n-2} + 2m)^{\frac{n-1}{n}}} \to_m 3^{\frac{1-n}{n}}.$$

- Freyer, H. $c(n) \geqslant 4^{-(n+o(n))}$ and for the class of unconditional bodies $c(n) \geqslant 3^{-n}$.

Some details

- Discrete John-type theorem.
  Berg, H., 2018. For $K \in \mathcal{K}_o^n$ there exists a $\mathbb{Z}^n$-basis A and $\boldsymbol{b} \in \mathbb{R}^n_{\geqslant 0}$ such that for $P(A, \boldsymbol{b}) = \{A\boldsymbol{z} : \boldsymbol{z} \in \mathbb{Z}^n, |z_i| \leqslant b_i, 1 \leqslant i \leqslant n\}$ holds

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  In particular,

$$\prod_{i=1}^{n}(2\lfloor b_i \rfloor + 1) \leqslant \mathrm{G}(K) \leqslant n^{O(\ln n)n} \prod_{i=1}^{n}(2\lfloor b_i \rfloor + 1).$$

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  - Hence

$$\begin{aligned}\mathrm{G}(mK) &= \sum_{\boldsymbol{s} \in \{0,\dots,m-1\}^n} \#(mK \cap (\boldsymbol{s} + m\mathbb{Z}^n)) \\ &= \sum_{\boldsymbol{t} \in \{0,1/m,\dots,1-1/m\}^n} \mathrm{G}(K - \boldsymbol{t}) \leqslant 2^n m^n \mathrm{G}(K).\end{aligned}$$

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- Let $\boldsymbol{a}_j = A\boldsymbol{e}_j \notin \boldsymbol{e}_j^{\perp}$, $1 \leqslant j \leqslant n-1$. Then

$$\mathrm{G}(K_j + \{-\lfloor b_j \rfloor \boldsymbol{a}_j, \ldots, \lfloor b_j \rfloor \boldsymbol{a}_j\}) = \mathrm{G}(K_j)\,(2\lfloor b_j \rfloor + 1)$$
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- Discrete Reverse Loomis-Whitney inequality?
  - Freyer, H. Let $K \in \mathcal{K}_o^n$, $\dim K \cap \mathbb{Z}^n = n$. Then there exist linearly independent $\boldsymbol{v}_i \in K \cap \mathbb{Z}^n$, $1 \leqslant i \leqslant n$, such that

$$\mathrm{G}(K)^{\frac{n-1}{n}} \geqslant c^{-n} \left(\prod_{i=1}^{n} \#(K|\boldsymbol{v}_i^{\perp} \cap \mathbb{Z}^n|\boldsymbol{v}_i^{\perp}) \right)^{\frac{1}{n}}.$$

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- Ingredient of the proof:
Discrete Brunn theorem: Let $K \in \mathcal{K}_o^n$ and let L be a k-dimensional linear lattice subspace. Then for $\boldsymbol{t} \in \mathbb{R}^n$

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- Let $K = \operatorname{conv}\{\pm([0,1]^{n-1} \times \{1\})\}$ and $L = \cap_{j=k+1}^{n} \boldsymbol{e}_j^{\perp}$. Then $\mathrm{G}(K \cap (\boldsymbol{e}_n + L)) = 2^{k-1}$ and $\mathrm{G}(K \cap L) = 1$.

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- inhomogeneous version:

$$\mathrm{G}(K)^{\frac{n-1}{n}} \leqslant O(n^{5/2}) \left(\prod_{i=1}^{n} \max\{\mathrm{G}(K \cap (\boldsymbol{a}_i^{\perp} + \boldsymbol{t})) : \boldsymbol{t} \in \mathbb{R}^n\} \right)^{\frac{1}{n}}.$$

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$$\mathrm{G}(K) \leqslant (\max\{\mathrm{G}(K \cap L) : L \in \mathcal{L}_{\mathbb{Z}}(1,n)\} + 1)^n - 2^n + 1$$
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  - Rabinowitz, 1989.

$$\mathrm{G}(K)^{\frac{1}{n}} \leqslant \max\{\mathrm{G}(K \cap A) : A \in \mathcal{A}_{\mathbb{Z}}(1,n)\}.$$

All bounds are tight.

- Alexander, H., Zvavitch, 2015. $K \in \mathcal{K}_o^n$.

$$\mathrm{G}(K)^{\frac{m}{n}} \leqslant O(1)^n n^{n-m} \max\{\mathrm{G}(K \cap L) : L \in \mathcal{L}_{\mathbb{Z}}(m, n)\}$$

- Alexander, H., Zvavitch, 2015. $K \in \mathcal{K}_o^n$.

$$\mathrm{G}(K)^{\frac{m}{n}} \leqslant O(1)^n n^{n-m} \max\{\mathrm{G}(K \cap L) : L \in \mathcal{L}_{\mathbb{Z}}(m, n)\}$$

- Freyer, H.

$$\mathrm{G}(K)^{\frac{m}{n}} \leqslant n^{O(n-m)} \max\{\mathrm{G}(K \cap A) : A \in \mathcal{A}_{\mathbb{Z}}(m, n)\}$$

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- Freyer, H.

$$\mathrm{G}(K)^{\frac{m}{n}} \leqslant n^{O(n-m)} \max\{\mathrm{G}(K \cap A) : A \in \mathcal{A}_{\mathbb{Z}}(m,n)\}$$

- Koldobsky, 2014; Chasapis, Giannopoulos, Liakopoulos, 2017. $\mathbf{0} \in \operatorname{int} K$.

$$\operatorname{vol}(K)^{\frac{m}{n}} \leqslant O(m)^{(n-m)/2} \max\{\operatorname{vol}_m(K \cap H) : H \in \mathcal{L}_{\mathbb{R}}(m,n)\}.$$

Thank you for your attention!