On the Comparison of Measures of Convex Bodies via Projections and Sections

Johannes Hosle

University of California, Los Angeles

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Notation

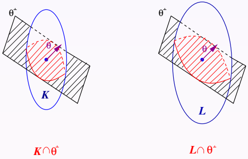
- $\mathbb{R}^n$ denotes the standard $n-$dimensional Euclidean space.
- Given $x = (x_1, ..., x_n) \in \mathbb{R}^n$, $|x| = (x_1^2 + ... + x_n^2)^{\frac{1}{2}}$ is the norm of x.
- $B_2^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$ is the unit ball in $\mathbb{R}^n$.
- $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ is the unit sphere in $\mathbb{R}^n$.
- A convex body is a compact, convex set in Euclidean space with nonempty interior.
- Given a convex body K in $\mathbb{R}^n$, $|K|$ denotes the Lebesgue measure of K.
- ω_n denotes $|B_2^n|$.
- K is in John's position if the unique ellipsoid of maximal volume contained within it is the unit ball.

The Busemann-Petty problem

If K, L are origin-symmetric convex bodies in $\mathbb{R}^n$ with

$$|K \cap \theta^\perp| \leq |L \cap \theta^\perp|$$

for all $\theta \in S^{n-1}$, does it follow that $|K| \leq |L|$?



Answer

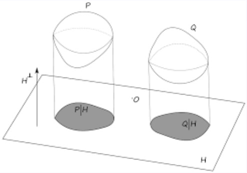
- Yes, if $n \leq 4$ and no if $n > 4$ (Gardner, Koldobsky, Schlumprecht; Zhang; Papadimitrakis).
- $|K| \leq cL_K|L|$ where L_K is the isotropic constant of K (Milman and Pajor).
- Best currently known bound on L_K is $cn^{\frac{1}{4}}$ (Bourgain; Klartag; Lee-Vempala).

The Shephard problem

If K, L are origin-symmetric convex bodies in $\mathbb{R}^n$ with

$$|K|\theta^{\perp}| \leq |L|\theta^{\perp}|$$

for all $\theta \in S^{n-1}$, does it follow that $|K| \leq |L|$?



Answer

- Yes, if $n \leq 2$ and no if $n > 2$ (Petty; Schneider; Koldobsky, Ryabogin, Zvavitch).
- $|K| \leq (1 + o(1))\sqrt{n}|L|$ and this bound is optimal (Ball).

V. Milman's variant of the Busemann-Petty and Shephard problem

If K, L are origin-symmetric convex bodies in $\mathbb{R}^n$ with

$$|K|\theta^\perp| \leq |L \cap \theta^\perp|$$

for all $\theta \in S^{n-1}$, does it follow that $|K| \leq |L|$?

- Hypotheses are stronger than those of the Busemann-Petty and Shephard problems.

Answer

Yes! (Giannopoulos and Koldobsky)

Reversal of Milman's question

If K, L are origin-symmetric convex bodies in $\mathbb{R}^n$ such that

$$|K \cap \theta^{\perp}| \leq |L|\theta^{\perp}|$$

for all $\theta \in S^{n-1}$, how can we compare $|K|$ and $|L|$?

- Hypotheses are weaker than those of the Busemann-Petty and Shephard problems.
- We cannot conclude $|K| \leq |L|$ for dimensions $n > 2$ by the solution of the Shephard problem. But even if $n = 2$ we can cannot conclude this inequality, as can be shown by a perturbation argument.

First question of the talk

Theorem

Let K, L be origin-symmetric convex bodies in $\mathbb{R}^n$ such that

$$|K \cap \theta^{\perp}| \leq |L|\theta^{\perp}|$$

for all $\theta \in S^{n-1}$. If $K \subset RB_2^n$ and $rB_2^n \subset L$, then

$$|K| \leq \frac{R}{r}|L|.$$

Corollary

If K and L are in John's position, then $|K| \leq \sqrt{n}|L|$.

- Let $\rho_K : S^{n-1} \to \mathbb{R}_+$ denote the radial function of K defined by $\rho_K(\theta) = \max\{t \geq 0 : t\theta \in K\}$.
- Polar coordinates:

$$|K| = \frac{1}{n} \int_{S^{n-1}} \rho_K^n(\theta) d\theta.$$

- The $(n-1)-$dimensional version of this formula is

$$|K \cap \xi^{\perp}| = \frac{1}{n-1} \int_{S^{n-1} \cap \xi^{\perp}} \rho_K^{n-1}(\theta) d\theta$$

  for any $\xi \in S^{n-1}$.
- To relate these formulas we shall use the following formula valid for all continuous f on the sphere:

$$\int_{G_{n,k}} \left(\int_{S^{n-1} \cap H} f(\xi) d\xi \right) d\nu_{n,k}(H) = \frac{|S^{k-1}|}{|S^{n-1}|} \int_{S^{n-1}} f(\xi) d\xi,$$

  where $\nu_{n,k}$ denotes the Haar probability measure on the Grassmanian $G_{n,k}$.

Proof of the theorem

Proof.

- Using $K \subset RB_2^n$ and the formulas on the previous slide,

$$\begin{aligned}
|K| &= \frac{1}{n}\int_{S^{n-1}} \rho_K^n(\theta)d\theta \\
&\leq \frac{R}{n}\int_{S^{n-1}} \rho_K^{n-1}(\theta)d\theta \\
&= \frac{R}{n|S^{n-2}|}\int_{S^{n-1}} \left(\int_{S^{n-1}\cap\xi^\perp} \rho_K^{n-1}(\theta)d\theta\right) d\xi \\
&= \frac{R}{n\omega_{n-1}}\int_{S^{n-1}} |K\cap\xi^\perp| d\xi.
\end{aligned}$$

Proof of the theorem

Proof.

- Since $|K\cap\theta^\perp| \le |L|\theta^\perp|$ for all $\theta\in S^{n-1}$,

$$|K| \le \frac{R}{n\omega_{n-1}}\int_{S^{n-1}}|K\cap\theta^\perp|d\theta \le \frac{R}{n\omega_{n-1}}\int_{S^{n-1}}|L|\theta^\perp|d\theta.$$

- Cauchy's surface area formula tells us that $|\partial L| = \frac{1}{\omega_{n-1}}\int_{S^{n-1}}|L|\theta^\perp|d\theta$ and therefore

$$|K| \le \frac{R}{n}|\partial L|.$$

- Since $rB_2^n\subset L$, we have

$$|\partial L| = \liminf_{\varepsilon\to 0}\frac{|L+\varepsilon rB_2^n|-|L|}{r\varepsilon} \le \liminf_{\varepsilon\to 0}\frac{|L(1+\varepsilon)|-|L|}{r\varepsilon} = \frac{n|L|}{r}.$$

- Therefore $|K|\le\frac{R}{r}|L|$ as desired.

Alternative estimate

Proposition

Our above assumptions also imply

$$|K| \leq cL_K^{\frac{1}{2}} n^{\frac{3}{4}} \left(\frac{R}{r}\right)^{\frac{n}{2n-1}} |L|.$$

Proof.

- Define the parallel section function $A_{K,\theta}(t) = |K \cap \{\theta^{\perp} + t\theta\}|$.
- By Fubini,

$$|K| = \int_{-R}^{R} A_{K,\theta}(t)dt.$$

- Since K is origin-symmetric, $A_{K,\theta}(t)$ is maximized for $t = 0$, and so

$$\begin{aligned} |K| &\leq 2R \min_{\theta \in S^{n-1}} |K \cap \theta^{\perp}| \\ &\leq 2R \min_{\theta \in S^{n-1}} |L|\theta^{\perp}| \\ &\leq cR\sqrt{n}|L|^{\frac{n-1}{n}}. \end{aligned}$$

Proof.

- Milman and Pajor proved that

$$|K|^{\frac{n-1}{n}} \leq cL_K \max_{\theta \in S^{n-1}} |K \cap \theta^{\perp}|.$$

- Therefore,

$$\begin{aligned} |K|^{\frac{n-1}{n}} &\leq cL_K \max_{\theta \in S^{n-1}} |L|\theta^{\perp}| \\ &\leq cL_K |\partial L| \\ &\leq \frac{cL_K n}{r} |L|. \end{aligned}$$

- Multiplying the two bounds gives

$$|K||K|^{\frac{n-1}{n}} \leq \frac{cL_K R n^{\frac{3}{2}}}{r} |L|^{\frac{n-1}{n}} |L|,$$

which implies

$$|K| \leq cL_K^{\frac{1}{2}} n^{\frac{3}{4}} \left(\frac{R}{r}\right)^{\frac{n}{2n-1}} |L|.$$

Definition

Given μ an absolutely continuous measure and K a convex body, we can define

$$P_{\mu,K}(\theta) = \frac{n}{2}\int_0^1 \mu_1(tK,[-\theta,\theta])dt,$$

where $\mu_1(A,B)$ is the mixed $\mu-$measure of A and B,

$$\mu_1(A,B) = \liminf_{\varepsilon\to 0}\frac{\mu(A+\varepsilon B)-\mu(A)}{\varepsilon}.$$

- This is a natural generalization of the formula

$$|K|\theta^{\perp}| = \frac{1}{2}\liminf_{\varepsilon\to 0}\frac{|K+\varepsilon[-\theta,\theta]|-|K|}{\varepsilon}$$

  for Lebesgue measure.
- Livshyts introduced this notion and proved a version of the Shephard problem for measures with a positive degree of concavity and homogeneity.

Theorem

Let μ be a log-concave measure with continuous ray-decreasing g. Assume that K, L are origin-symmetric convex bodies in $\mathbb{R}^n$ such that

$$P_{\mu,K}(\theta) \leq \mu_{n-1}(L \cap \theta^{\perp})$$

for all $\theta \in S^{n-1}$. Let $r > 0$ be a fixed-parameter.
(a) If $\frac{1}{e}\mu(rB_2^n) \leq \mu(K) \leq \mu(rB_2^n)$, then

$$\mu(K) \log \frac{\mu(rB_2^n)}{\mu(K)} \leq r\omega^{\frac{1}{n}} \|g\|_{\infty}^{\frac{1}{n}} \mu(L)^{\frac{n-1}{n}}.$$

(b) If $\mu(K) \leq \frac{1}{e}\mu(rB_2^n)$, then

$$\mu(K) \leq \left(\frac{er^n \omega_n \|g\|_{\infty}}{\mu(rB_2^n)} \right)^{\frac{1}{n-1}} \mu(L).$$

- Recall that a measure μ is log-concave if for all measurable K, L and $\lambda \in [0,1]$ we have

$$\mu((1-\lambda)K + \lambda L) \geq \mu(K)^{1-\lambda}\mu(L)^{\lambda}.$$

- A function f is called log-concave if $\log f$ is concave. A measure with a log-concave density is log-concave.

Definition

A density $g : \mathbb{R}^n \to [0,\infty)$ is ray-decreasing if $g(tx) \geq g(x)$ for all $t \in [0,1]$ and $x \in \mathbb{R}^n$.

Log-concave lemma

Let μ be a log-concave measure and E, F be measurable sets. Then

$$\mu_1(E, F) \geq \mu_1(E, E) + \mu(E) \log \frac{\mu(F)}{\mu(E)}.$$

Ray-decreasing lemma

Let μ be a measure with a ray-decreasing density g.

- If $t \in [0, 1]$ and K is measurable, $\mu(tK) \geq t^n \mu(K)$.
- Moreover, we have the limits

$$\lim_{s \to \infty} \frac{\mu_1(sB_1^n, B_2^n)}{\mu(sB_2^n)} = 0, \lim_{s \to 0} \frac{\mu_1(sB_2^n, B_2^n)}{\mu(sB_2^n)} = \infty.$$

Theorem (Dann, Paouris, Pivovarov)

Let $1 \leq k \leq n-1$ and f be a nonnegative, bounded, integrable function on $\mathbb{R}^n$. Then

$$\int_{G_{n,k}} \frac{\left(\int_E f(x)dx\right)^n}{\|f|E\|_\infty^{n-k}} d\nu_{n,k}(E) \leq \frac{\omega_k^n}{\omega_n^k} \left(\int_{\mathbb{R}^n} f(x)dx\right)^k.$$

- $\|f|E\|_\infty$ is the L^∞−norm of f restricted to the k−dimensional subspace E.

Proof of the theorem, Part (a)

Proof of (a).

- For $t \in [0,1]$ and $s > 0$ to be chosen later, the Log-concave lemma tells us that

$$\begin{aligned}\mu_1(tK, B_2^n) &\geq \mu_1(tK, sB_2^n)\\ &\geq \mu_1(tK, tK) + \mu(tK)\log\frac{\mu(sB_2^n)}{\mu(tK)}\\ &= t\frac{d}{dt}\mu(tK) + \mu(tK)\log\frac{\mu(sB_2^n)}{\mu(tK)}.\end{aligned}$$

- Integrate both sides in t from 0 to 1 to get

$$\mu(K) \leq s\int_0^1 \mu_1(tK, B_2^n)dt + \int_0^1 \mu(tK)\log\frac{e\mu(tK)}{\mu(sB_2^n)}dt.$$

Proof of (a).

- Using Parseval's formula on the sphere, an analog of the Cauchy surface area formula for measures can be proven:

$$\int_0^1 \mu_1(tK, B_2^n)dt = \frac{1}{n\omega_{n-1}} \int_{S^{n-1}} P_{\mu,K}(u)du.$$

- Thus,

$$\mu(K) \leq \frac{s}{n\omega_{n-1}} \int_{S^{n-1}} P_{\mu,K}(u)du + \int_0^1 \mu(tK) \log \frac{e\mu(tK)}{\mu(sB_2^n)} dt$$

Proof of (a).

- Using our assumption $P_{\mu,K}(\theta) \leq \mu_{n-1}(L \cap \theta^{\perp})$ along with Jensen's inequality, we have

$$\frac{1}{n\omega_{n-1}} \int_{S^{n-1}} P_{\mu,K}(u)du \leq \frac{1}{n\omega_{n-1}} \int_{S^{n-1}} \mu_{n-1}(L \cap \theta^{\perp})d\theta$$
$$= \frac{\omega_n}{\omega_{n-1}} \int_{S^{n-1}} \mu_{n-1}(L \cap \theta^{\perp})d\sigma(\theta)$$
$$\leq \frac{\omega_n}{\omega_{n-1}} \left(\int_{S^{n-1}} \mu_{n-1}(L \cap \theta^{\perp})^n d\sigma(\theta) \right)^{\frac{1}{n}},$$

where $\sigma(\theta) = d\frac{\theta}{|S^{n-1}|}$ is the normalized probability measure on the sphere.

Proof of the theorem, Part (a)

Proof of (a).

- By the theorem of Dann, Paouris, and Pivovarov, we write

$$\left(\int_{S^{n-1}} \mu_{n-1}(L\cap\theta^{\perp})^n d\sigma(\theta)\right)^{\frac{1}{n}} = \left(\int_{S^{n-1}} \left(\int_{\theta^{\perp}} g(x)\chi_L(x)dx\right)^n d\sigma(\theta)\right)^{\frac{1}{n}}$$
$$\leq \left(\|g\|_{\infty} \frac{\omega_{n-1}^n}{\omega_n^{n-1}}\right)^{\frac{1}{n}} \mu(L)^{\frac{n-1}{n}}.$$

- Therefore,

$$\mu(K) \leq s\omega_n^{\frac{1}{n}} \|g\|_{\infty}^{\frac{1}{n}} \mu(L)^{\frac{n-1}{n}} + \int_0^1 \mu(tK)\log e\mu(tK)dt$$
$$+ \left(\int_0^1 \mu(tK)dt\right) \log\frac{1}{\mu(sB_2^n)}.$$

Proof of (a).

- We now choose s to optimize the previous inequality, namely

$$\frac{\mu_1(sB_2^n, B_2^n)}{\mu(sB_2^n)} = \frac{\omega_n^{\frac{1}{n}} \|g\|_\infty^{\frac{1}{n}} \mu(L)^{\frac{n-1}{n}}}{\int_0^1 \mu(tK)dt}.$$

- This s is guaranteed to exist by the Ray-decreasing lemma.
- With this choice of s and with r as in the statement of the theorem, we may bound

$$\log \frac{1}{\mu(sB_2^n)} \leq \frac{\omega_n^{\frac{1}{n}} \|g\|_\infty^{\frac{1}{n}} \mu(L)^{\frac{n-1}{n}}}{\int_0^1 \mu(tK)dt}(r-s) - \log \mu(rB_2^n)$$

using the Log-concave lemma.
- Thus,

$$\mu(K) \leq r\omega_n^{\frac{1}{n}} \|g\|_\infty^{\frac{1}{n}} \mu(L)^{\frac{n-1}{n}} + \int_0^1 \mu(tK) \log \frac{e\mu(tK)}{\mu(rB_2^n)} dt.$$

Proof of (a).

- From Jensen's inequality, we derive the bound

$$\mu(K) \leq r\omega_n^{\frac{1}{n}} \|g\|_\infty^{\frac{1}{n}} \mu(L)^{\frac{n-1}{n}} + \log\max\left(1, \left(\frac{e\mu(K)}{\mu(rB_2^n)}\right)^{\mu(K)}\right).$$

- Recall that the assumption of Part (a) is that $\mu(K) \geq \frac{1}{e}\mu(rB_2^n)$.
- Therefore,

$$\mu(K)\log\frac{\mu(rB_2^n)}{\mu(K)} \leq r\omega_n^{\frac{1}{n}} \|g\|_\infty^{\frac{1}{n}} \mu(L)^{\frac{n-1}{n}},$$

finishing the proof of Part (a).

Proof of the theorem, Part (b)

Restatement of the theorem

Let μ be a log-concave measure with continuous ray-decreasing g. Assume that K, L are origin-symmetric convex bodies in $\mathbb{R}^n$ such that

$$P_{\mu,K}(\theta) \leq \mu_{n-1}(L \cap \theta^{\perp})$$

for all $\theta \in S^{n-1}$. Let $r > 0$ be a fixed-parameter.
(b) If $\mu(K) \leq \frac{1}{e}\mu(rB_2^n)$, then

$$\mu(K) \leq \left(\frac{er^n \omega_n \|g\|_\infty}{\mu(rB_2^n)} \right)^{\frac{1}{n-1}} \mu(L).$$

Proof of (b).

- Since $\mu(K) \leq \frac{1}{e}\mu(rB_2^n)$, for every $t \in [0,1]$ there exists $f(t) \in [0,1]$ such that $\mu(rf(t)B_2^n) = e\mu(tK)$.
- Following the same setup as part (a), we have

$$\mu(K) \leq r \int_0^1 f(t)\mu_1(tK, B_2^n) + \mu(tK) \log \frac{e\mu(tK)}{\mu(rf(t)B_2^n)} dt.$$

Proof of (b).

- By the choice of f, this inequality becomes

$$\mu(K) \leq r \int_0^1 f(t)\mu_1(tK, B_2^n)dt.$$

- Moreover, by the Ray-decreasing lemma,

$$f(t)^n \mu(rB_2^n) \leq \mu(rf(t)B_2^n) = e\mu(tK) \leq e\mu(K),$$

and so

$$\mu(K) \leq r\left(\frac{e\mu(K)}{\mu(rB_2^n)}\right)^{\frac{1}{n}} \int_0^1 \mu_1(tK, B_2^n)dt \leq r\left(\frac{e\mu(K)}{\mu(rB_2^n)}\right)^{\frac{1}{n}} \omega_n^{\frac{1}{n}} \|g\|_\infty^{\frac{1}{n}} \mu(L)^{\frac{n-1}{n}},$$

which rearranges to

$$\mu(K) \leq \left(\frac{er^n\omega_n \|g\|_\infty}{\mu(rB_2^n)}\right)^{\frac{1}{n-1}} \mu(L).$$

- The Loomis-Whitney inequality states that if $u_1, \ldots, u_n$ form an orthonormal basis of $\mathbb{R}^n$ and K is a convex body in $\mathbb{R}^n$, then

$$|K|^{n-1} \leq \prod_{i=1}^{n} |K|u_i^{\perp}|,$$

  with equality if and only if K is a box with faces parallel to the hyperplanes $u_i^{\perp}$.

- This was extended by Ball, who showed that $u_1, \ldots, u_m \in \mathbb{R}^m$ and $c_1, \ldots, c_m$ are positive constants such that

$$\sum_{i=1}^{m} c_i u_i \otimes u_i = I_n,$$

  then

$$|K|^{n-1} \leq \prod_{i=1}^{m} |K|u_i^{\perp}|^{c_i}.$$

Positively concave and positively homogeneous measures

- A function $f : \mathbb{R}^n \to [0, \infty]$ is $p-$concave if f^p is concave on the support of f.
- A function $f : \mathbb{R}^n \to [0, \infty]$ is $r-$homogeneous if $f(ax) = a^r f(x)$ for all $a > 0$ and $x \in \mathbb{R}^n$.
- We will be interested in functions g that are both $s-$concave for some $s > 0$ and $\frac{1}{p}-$homogeneous for some $p > 0$. Such functions will necessarily be $p-$concave. Moreover, with the exception of the constant functions, all such g will be supported on convex cones. E.g. $g(x) = 1_{\langle x, \theta \rangle > 0} \langle x, \theta \rangle^{\frac{1}{p}}$.
- $\tilde{g}(x) = g(x) + g(-x)$.
- Measures with such densities were studied by Milman and Rotem.

Lemma (Borell)

Let μ be a measure with a $p-$concave density g. Then, for $q = \frac{1}{n+\frac{1}{p}}$, μ is $q-$concave, that is for measurable E, F and $\lambda \in [0,1]$ we have

$$\mu(\lambda E + (1-\lambda)F) \geq (\lambda\mu(E)^q + (1-\lambda)\mu(F)^q)^{\frac{1}{q}}.$$

- Moreover, by a change of variables, if μ has $\frac{1}{p}-$homogeneous density, then μ is $\frac{1}{q}-$homogeneous, that is

$$\mu(tE) = t^{\frac{1}{q}}\mu(E)$$

for $t > 0$.

Theorem

Let μ be a measure with a $p-$concave, $\frac{1}{p}-$homogeneous density g for some $p>0$. Then, for any convex body K and an orthonormal basis $(u_i)_{i=1}^n$ with $[-u_i, u_i] \cap \text{supp}(g) \neq \varnothing$ for each $1 \leq i \leq n$,

$$\mu(K)^{n+\frac{1}{p}-1} \leq 2^{n+\frac{1}{p}} \left(1+\frac{1}{pn}\right)^n \left(\sum_{k=1}^n \tilde{g}^p(u_k)\right)^{-\frac{1}{p}} \prod_{i=1}^n P_{\mu,K}(u_i)^{1+\frac{\tilde{g}^p(u_i)}{p\sum_{k=1}^n \tilde{g}^p(u_k)}}.$$

- Recall that $P_{\mu,K}(\theta) = \frac{n}{2}\int_0^1 \mu_1(tK, [-\theta,\theta])dt$.
- We remark that a similar extension can also be proven for Ball's inequality.

Outline of the proof

- Take the box $Z = \sum_{i=1}^{n} \alpha_i[-u_i, u_i]$ with $\alpha_i = \frac{1}{P_{\mu,K}(u_i)}$.
- We use Minkowski's first inequality (for $q-$concave measures) to write

$$\mu(K)^{1-q} \leq q\mu(Z)^{-q}\mu_1(K,Z) = 2\mu(Z)^{-q}.$$

- Without loss of generality, $u_i \in \text{supp}(g)$ and $g(-u_i) = 0$ for all $1 \leq i \leq n$. Let us define F_i to be the face of Z orthogonal to and touching $\alpha_i u_i$.
- By homogeneity,

$$\mu(Z) = q\sum_{i=1}^{n} \alpha_i \mu_{n-1}(F_i),$$

  where $\mu_{n-1}(F_i)$ denotes the integral of g over the $(n-1)-$dimensional set F_i.
- It remains to find an appropriate lower bound for $\mu_{n-1}(F_i)$.

Lemma

Let $g, \mu, (u_i)_{i=1}^n, F_i$ be as above. Then,

$$\mu_{n-1}(F_i) \geq \left(\frac{pn}{pn+1}\right)^n \left(1 + \frac{\tilde{g}^p(u_i)}{p\sum_{k=1}^n \tilde{g}^p(u_k)}\right)\left(\sum_{i=1}^n \tilde{g}^p(u_i)\right)^{\frac{1}{p}}$$

$$\times \alpha_i^{-1} \prod_{j=1}^n \alpha_j^{1+\frac{\tilde{g}^p(u_j)}{p\sum_{i=1}^n \tilde{g}^p(u_i)}}.$$

Proof of lemma.

- Without of loss of generality, we consider $i = 1$.
- We begin by writing $\mu_{n-1}(F_1)$ as an integral of g over F_1, subdividing the domain of integration, and using homogeneity:

$$\mu_{n-1}(F_1) := \int_{\substack{v = \alpha_1 u_1 + \sum_{j=2}^{n} \beta_j u_j \\ |\beta_j| \leq \alpha_j}} g(v) dv$$

$$= \sum_{\sigma=(\pm1,\ldots,\pm1)} \int_0^{\alpha_n} \ldots \int_0^{\alpha_2} g\left(\alpha_1 u_1 + \sum_{j=2}^{n} \beta_j \sigma(j) u_j\right) d\beta_2 \ldots d\beta_n$$

$$= \sum_{\sigma=(\pm1,\ldots,\pm1)} \int_0^{\alpha_n} \ldots \int_0^{\alpha_2} \left(\alpha_1 + \sum_{j=2}^{n} \beta_j\right)^{\frac{1}{p}}$$

$$\times g\left(\frac{\alpha_1}{\alpha_1 + \sum_{j=2}^{n} \beta_j} u_1 + \sum_{j=2}^{n} \frac{\beta_j}{\alpha_1 + \sum_{j=2}^{n} \beta_j} \sigma(j) u_j\right) d\beta_2 \ldots d\beta_n.$$

Proof of lemma.

- Since $u_i \in \mathrm{supp}(g)$ for $1 \leq i \leq n$, we can only use concavity to estimate from below the integral where σ is the identity permutation. This accounts for the factor of 2^n in the Theorem.
- We have the inequality

$$\mu_{n-1}(F_1) \geq \int_0^{\alpha_n} \ldots \int_0^{\alpha_2} \left(\alpha_1 + \sum_{j=2}^{n} \beta_j \right)^{\frac{1}{p}} \times g\left(\frac{\alpha_1}{\alpha_1 + \sum_{j=2}^{n} \beta_j} u_1 + \sum_{j=2}^{n} \frac{\beta_j}{\alpha_1 + \sum_{j=2}^{n} \beta_j} u_j \right) d\beta_2 \ldots d\beta_n.$$

Proof of lemma.

- By p-concavity of g,

$$\mu_{n-1}(F_1) \geq \int_0^{\alpha_n} \dots \int_0^{\alpha_2} \left(\alpha_1 + \sum_{j=2}^{n} \beta_j\right)^{\frac{1}{p}}$$

$$\times \left(\frac{\alpha_1}{\alpha_1 + \sum_{j=2}^{n} \beta_j} g^p(u_1) + \sum_{j=2}^{n} \frac{\beta_j}{\alpha_1 + \sum_{j=2}^{n} \beta_j} g^p(u_j)\right)^{\frac{1}{p}} d\beta_2 \dots d\beta_n$$

$$= \int_0^{\alpha_n} \dots \int_0^{\alpha_2} \left(\alpha_1 g^p(u_1) + \sum_{j=2}^{n} \beta_j g^p(u_j)\right)^{\frac{1}{p}} d\beta_2 \dots d\beta_n$$

$$= \left(\sum_{i=1}^{n} g^p(u_i)\right)^{\frac{1}{p}} \int_0^{\alpha_n} .. \int_0^{\alpha_2} \left(\alpha_1 \frac{g^p(u_1)}{\sum_{i=1}^{n} g^p(u_i)} + \sum_{j=2}^{n} \beta_j \frac{g^p(u_j)}{\sum_{i=1}^{n} g^p(u_i)}\right)^{\frac{1}{p}}$$

$d\beta_2 .. d\beta_n$.

Proof of lemma.

- From the arithmetic-mean geometric-mean inequality,

$$\alpha_1 \frac{g^p(u_1)}{\sum_{i=1}^n g^p(u_i)} + \sum_{j=2}^n \beta_j \frac{g^p(u_j)}{\sum_{i=1}^n g^p(u_i)} \geq \alpha_1^{\frac{g^p(u_1)}{\sum_{i=1}^n g^p(u_i)}} \prod_{j=2}^n \beta_j^{\frac{g^p(u_j)}{\sum_{i=1}^n g^p(u_i)}}.$$

- Substituting this product under the integral, evaluating, and applying one more arithmetic-mean geometric-mean inequality, we conclude the proof of the lemma and the theorem.

Thanks for your attention!