On non-central sections of the simplex, the cube and the cross-polytope

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Let $K \subset \mathbb{R}^n$ be a symmetric convex body, $a \in S^{n-1} \subset \mathbb{R}^n$ and $t \in \mathbb{R}$.

$$A(a,t) := vol_{n-1}(\{x \in K \mid \langle x, a \rangle = t\}) \quad \textit{parallel section function},$$

$$P(a,t) := vol_{n-2}(\{x \in \partial K \mid \langle x, a \rangle = t\}) \quad \textit{perimeter function}.$$

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Non-central sections for $t > 0$.

Theorem 1

(a) $K = [-\frac{1}{2}, \frac{1}{2}]^n$, $d := \frac{\sqrt{n-1}}{2} < t \leq \frac{\sqrt{n}}{2}$, $a^{(n)} := \frac{1}{\sqrt{n}}(1, \dots, 1)$. *Then we have for all* $a \in S^{n-1}$ *that* $A(a,t) \leq A(a^{(n)}, t)$.

(b) $K = B(l_1^n)$, $d := \frac{1}{\sqrt{2}} < t \leq 1$, $e_1 := (1, 0, \dots, 0)$. *Then we have for all* $a \in S^{n-1}$ *that* $A(a,t) \leq A(e_1, t)$.

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(b) $K = B(l_1^n)$, $d := \frac{1}{\sqrt{2}} < t \leq 1$, $e_1 := (1, 0, \dots, 0)$. Then we have for all $a \in S^{n-1}$ that $A(a,t) \leq A(e_1, t)$.

(a) is due to **Moody, Stone, Zach and Zvavitch**, (b) is due to **Liu, Tkocz**. In both case, d is the distance of the midpoint of edges to 0. We first consider the corresponding problem for the simplex.

Let $\Delta^n := \{x \in \mathbb{R}_+^{n+1} \mid \sum_{j=1}^{n+1} x_j = 1\}$ be the n-dimensional simplex of side-length $\sqrt{2}$, $vol_n(\Delta^n) = \frac{\sqrt{n+1}}{n!}$. Then $c := \frac{1}{n+1}(1, \dots, 1)$ is the centroid of Δ^n. Let $a \in S^n \subset \mathbb{R}^{n+1}$ be such that $\sum_{j=1}^{n+1} a_j = 0$. Then $c \in a^\perp$. Similar as in the symmetric case we define

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Then t is the distance of the hyperplanes $[\langle x, a \rangle = 0]$ through c and $[\langle x, a \rangle = t]$, $d := \sqrt{\frac{n-1}{2(n+1)}}$ is the distance of the midpoint of edges of Δ^n to the centroid c and $D := \sqrt{\frac{n}{n+1}}$ is the distance of vertices of Δ^n to the centroid c.

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Let $a^{[n]} := (\sqrt{\frac{n}{n+1}}, -\frac{1}{\sqrt{n(n+1)}}, \dots, -\frac{1}{\sqrt{n(n+1)}}) \in S^n$ be the unit vector in the direction from c to the vertex e_1. Then $(a^{[n]})^\perp$ is a hyperplane through c parallel to a face.

Central simplex sections

Theorem 2

Let $K = \Delta^n$, $\tilde{a} := \frac{1}{\sqrt{2}}(1, -1, 0, \dots, 0)$. *Then for all* $a \in S^{n-1}$ *with* $\sum_{j=1}^{n+1} a_j = 0$

$$A(a, 0) \leq A(\tilde{a}, 0) = \frac{\sqrt{n+1}}{(n-1)!} \frac{1}{\sqrt{2}}.$$

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Thus the maximal volume hyperplanes through c pass through the midpoint of an edge and the other vertices. This theorem is due to **Webb**.

$a^{[n]}$ probably yields the minimal volume hyperplane through c, as claimed by Filliman (no published proof).

Non-central simplex sections

Theorem 3

Let $n \geq 3$, $K = \Delta^n$, $d := \sqrt{\frac{n-1}{2(n+1)}} < t \leq D := \sqrt{\frac{n}{n+1}}$ *and* $a^{[n]} := (\sqrt{\frac{n}{n+1}}, -\frac{1}{\sqrt{n(n+1)}}, \dots, -\frac{1}{\sqrt{n(n+1)}}) \in S^n$. *Then for all* $a \in S^{n-1}$ *with* $\sum_{j=1}^{n+1} a_j = 0$

$$A(a,t) \leq A(a^{[n]},t) = \frac{\sqrt{n+1}}{(n-1)!} \, (\frac{n}{n+1})^{n/2} \, (\sqrt{\frac{n}{n+1}} - t)^{n-1}.$$

For $n = 2$ *we have the same result, if* $\frac{5}{4}\frac{1}{\sqrt{6}} \leq t \leq \sqrt{\frac{2}{3}}$. *For* $\frac{1}{\sqrt{6}} < t < \frac{5}{4}\frac{1}{\sqrt{6}}$ *the statement does not hold.*

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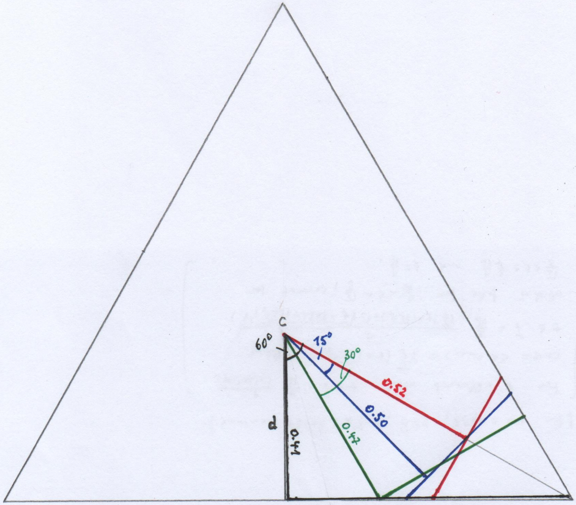
The hyperplanes of maximal volume at distance t to the centroid c are those parallel to faces.



C
60°
15°
30°
0.52
0.50
0.47
d
0.41

Idea of proof.

(a) $||a||_2 = 1$, $\sum_{j=1}^{n+1} a_j = 0$, $d(n) := \sqrt{\frac{n-1}{2(n+1)}} = ||c - \frac{e_1+e_j}{2}||_2 < t$. If $\{x \in \Delta^n \mid \langle x, a\rangle = t\}$ is non-trivial, $\langle a, e_i\rangle > t$ for some i, say $\langle a, e_1\rangle > t$. Then $\langle a, e_j\rangle < t$ for all other $j \in \{2, \dots, n+1\}$.

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Claim: $A(a,t) = \frac{\sqrt{n+1}}{(n-1)!} \prod_{j=2}^{n+1} \frac{1}{a_1 - a_j} (a_1 - t)^{n-1}$.
Let $v_j = s_j e_1 + (1 - s_j)e_j \in [\langle a, x\rangle = t] \cap span(e_1, e_j)$,
$v_j - e_1 = (1 - s_j)(e_j - e_1)$. Then $P := \{x \in \Delta^n \mid \langle a, x\rangle \geq t\}$ is a pyramid spanned by the vectors $v_j - e_1$, $j = 2, \dots, n+1$.

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$v_j - e_1 = (1 - s_j)(e_j - e_1)$. Then $P := \{x \in \Delta^n \mid \langle a, x\rangle \geq t\}$ is a pyramid spanned by the vectors $v_j - e_1$, $j = 2, \dots, n+1$. Hence

$$vol_n(P) = \frac{\sqrt{n+1}}{n!} \prod_{j=2}^{n+1} (1 - s_j) = \frac{\sqrt{n+1}}{n!} \prod_{j=2}^{n+1} \frac{a_1 - t}{a_1 - a_j},$$

yielding the formula for $A(a,t)$, since $a_1 - t =$ height of the pyramid P.
Only needed: $a_1 > t > a_j$, $j = 2, \dots, n+1$.

Claim: If $a \in S^n$ attains the maximum of $A(a,t)$ with $a_1 > t > d(n)$, we have $a_2 = \cdots = a_{n+1}$.
This, together with $\sum_{j=1}^{n+1} a_j = 0$ implies $a = a^{[n]}$. Hence for each vertex, there is a **unique** maximal hyperplane, which is parallel to a face.

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Use the Lagrange multiplier equations for
$f(a,t) := (n-1)\ln(a_1 - t) - \sum_{j=2}^{n+1} \ln(a_1 - a_j)$ with $||a||_2^2 = 1$, $\sum_{j=1}^{n+1} a_j = 0$ to show that the coordinates a_j satisfy a quadratic equation $a_j^2 - pa_j - q = 0$, with p, q independent of $j \in \{2, \dots, n+1\}$, the larger solution of which does not satisfy $a_j < t$ if $t > d(n)$.

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The only possible critical value then is $a^{[n]}$ which is a relative maximum.

Theorem 4

Let $n \geq 3$, $K = \Delta^n$, $-\frac{1}{\sqrt{n(n+1)}} < t < d(n)$, $c(n) = \frac{2n+1}{n(n+2)}\sqrt{\frac{n}{n+1}}$.
Then $a^{[n]}$ *is*
(a) a local maximum of $A(.,t)$ *if* $c(n) < t < d(n)$ *and*
(b) a local minimum of $A(.,t)$ *if* $-\frac{1}{\sqrt{n(n+1)}} < t < c(n)$.

In particular, $a^{[n]}$ yields a **local** minimum for the centroid section $A(.,0)$.

Note that $c(n)$ is of order $\frac{2}{n+1}$.

Perimeter of simplex sections

We have the formula for $P(a,t)$, if $a_1 > t$, $t > a_j$ for $j = 1, \dots, n+1$,

$$P(a,t) = \frac{1}{(n-2)!} \sum_{j=2}^{n+1} \sqrt{n-(n+1)a_j^2} \prod_{k=2, k\neq j}^{n+1} \frac{1}{a_1 - a_k} \, (a_1 - t)^{n-2} \, .$$

There is no term for $j = 1$ since $[\langle a, x\rangle = t]$ does not meet $[x_1 = 0]$.

Theorem 5

Let $n \geq 4$, $d(n) := \sqrt{\frac{n-1}{2(n+1)}} < t \leq a_1 \leq \sqrt{\frac{n}{n+1}}$. *Then for all* $a \in S^n \subset \mathbb{R}^{n+1}$ *with* $\sum_{j=1}^{n+1} a_j = 0$

$$P(a,t) \leq P(a^{[n]}, t) = \frac{n\sqrt{n-1}}{(n-2)!} \left(\frac{n}{n+1}\right)^{(n-2)/2} \left(\sqrt{\frac{n}{n+1}} - t\right)^{n-2} \, .$$

For $n = 4$, *we need to assume* $0.671 \simeq \frac{3}{\sqrt{20}} < t \leq \sqrt{\frac{4}{5}}$.

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Proposition 1

Assume $n \ge 4$, $c(n) = \frac{3n+2}{n(n+2)} \sqrt{\frac{n}{n+1}} < t < \sqrt{\frac{n-1}{2(n+1)}}$.
Then $a^{[n]}$ *is a local maximum of* $P(.,t)$.

Note that $c(n) \simeq \frac{3}{n+1}$.

The **central** hyperplane yielding the minimal perimeter of the simplex for $t = 0$ seems to depend on the dimension n. For $n = 3$ it is the section of the simplex Δ^3 by $\bar{a} = \frac{1}{2}(1, -1, 1, -1)$. In this case, $\{x \in \partial\Delta^3 \mid \langle \bar{a}, x \rangle = 0\}$ is a square of side-length $\frac{1}{\sqrt{2}}$, thus $P(\bar{a}, 0) = 2\sqrt{2}$, whereas for $a^{[3]} = (\frac{\sqrt{3}}{2}, -\frac{1}{2\sqrt{3}}, -\frac{1}{2\sqrt{3}}, -\frac{1}{2\sqrt{3}})$ we have $P(a^{[3]}, 0) = \frac{9}{4}\sqrt{2}$, when the section is a triangle. Therefore

$$A(\bar{a}, 0) = \frac{1}{2} > A(a^{[3]}, 0) = \frac{9}{32}\sqrt{3} \; , \; P(\bar{a}, 0) = 2\sqrt{2} < P(a^{[3]}, 0) = \frac{9}{4}\sqrt{2} \; .$$

Corresponding examples do not extend beyond dimension $n > 9$.

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Corresponding examples do not extend beyond dimension $n > 9$.

Question: Is it true that for all $n \geq 3$ and all $a \in S^n$ with $\sum_{j=1}^{n+1} a_j = 0$

$$P(a, 0) \leq P(\tilde{a}, 0) = \frac{\sqrt{n-1}}{(n-2)!}\ (\sqrt{\frac{n(n-1)}{2}} + 1)\ ,\ \ \tilde{a} = \frac{1}{\sqrt{2}}(1, -1, 0, \dots, 0)\ ?$$

This would be the perimeter analogue of Webb's result for the section area. I can prove at least

$$P(a, 0) \leq P(\tilde{a}, 0)(1 + \frac{1}{n})\ .$$

Parallel section function of the cross-polytope

For central sections of the l_1^n-ball, **Meyer, Pajor** showed

Theorem 6

Let $K = B(l_1^n)$. *Then for all* $a \in S^{n-1}$

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The maximal volume hyperplanes are orthogonal to the coordinate directions. For non-central sections of the l_1^n-ball, we have by **Liu, Tkocz**

Theorem 7

Let $K = B(l_1^n)$, $n \geq 3$, $\frac{1}{\sqrt{2}} < t \leq 1$ and $a \in S^{n-1} \subset \mathbb{R}^n$ with $a_1 > t > a_j$, $j = 2, \ldots, n$. Then

$$A(a,t) \leq A(e_1,t) = \frac{2^{n-1}}{(n-1)!}(1-t)^{n-1} \; .$$

For $n = 2$, one needs $\frac{3}{4} < t \leq 1$ for the same result.

Again the maximal volume hyperplanes are orthogonal to the coordinate directions.

Theorem 8

Let $n \geq 3$, $0 < t \leq \frac{1}{\sqrt{2}}$. Then e_1 is a **local** *maximum of the parallel section function $A(., t)$ of the cross-polytope, if $\frac{3}{n+2} < t \leq \frac{1}{\sqrt{2}}$ and a* **local** *minimum if $0 < t < \frac{3}{n+2}$. For $n = 2$, we have a local minimum for $0 < t < \frac{3}{4}$.*

An easy explicit example for $t = \frac{2}{n}$ is $\tilde{a} = (\frac{n-2}{n}, \frac{2}{n}, \dots, \frac{2}{n})$ with

$$A(\tilde{a}, \frac{2}{n}) > A(e_1, \frac{2}{n}).$$

.

Perimeter of sections of the cross-polytope.

Let $||a||_2 = 1$, $a_1 > t > a_j$, $j = 2, \dots, n$ and $t > \frac{1}{\sqrt{2}}$. Using the method of Liu, Tkocz one finds the formula for the perimeter

$$P(a,t) = \frac{\sqrt{n}}{(n-2)!} \sum_{\epsilon \in \{-1,1\}^n} \sqrt{1 - \frac{1}{n}\langle a, \epsilon\rangle^2} \prod_{j=2}^{n} \frac{1}{a_1 - \epsilon_j a_j} \, (a_1 - t)^{n-2} \; .$$

The distance from the center of the face $Conv(e_1, \epsilon_2 e_2, \dots, \epsilon_n e_n)$ to the intersection with the hyperplane is given by $t_\epsilon = \frac{t - \frac{1}{n}\langle a,\epsilon\rangle}{\sqrt{1 - \frac{1}{n}\langle a,\epsilon\rangle^2}}$.

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Theorem 9

We have for $n \geq 4$ and $t > \frac{1}{\sqrt{2}}$

$$P(a,t) \leq P(e_1,t) = \frac{\sqrt{n-1}}{(n-2)!} \, 2^{n-1} \, (1-t)^{n-2} \, .$$

For $n = 3$, this is true at least if $t > \frac{4}{5}$.

The proof relies on the Cauchy-Schwarz inequality, the log-convexity of $\frac{\sqrt{1+x}}{1-x}$ and differentiation techniques.
Numerical evidence: The result is true also for $n = 3$ and all $t > \frac{1}{\sqrt{2}}$.

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There is also a local maximum result for the perimeter for $n \geq 6$ and $t > \frac{4}{n}$.

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We have for $n \geq 4$ and $t > \frac{1}{\sqrt{2}}$

$$P(a,t) \leq P(e_1,t) = \frac{\sqrt{n-1}}{(n-2)!} \, 2^{n-1} \, (1-t)^{n-2} \, .$$

For $n = 3$, this is true at least if $t > \frac{4}{5}$.

The proof relies on the Cauchy-Schwarz inequality, the log-convexity of $\frac{\sqrt{1+x}}{1-x}$ and differentiation techniques.
Numerical evidence: The result is true also for $n = 3$ and all $t > \frac{1}{\sqrt{2}}$.
There is also a local maximum result for the perimeter for $n \geq 6$ and $t > \frac{4}{n}$.

As mentioned, **Meyer, Pajor** showed that for all $a \in S^{n-1}$, $A(a,0) \leq A(e_1,0)$. For the perimeter of the cross-polytope there is at least the asymptotic estimate for all $a \in S^{n-1}$

$$P(a,0) \leq \left(\frac{n}{n-1}\right)^{1/2} P(e_1,0) \, .$$

Cubic sections

Let $Q_n := [-\frac{1}{2}, \frac{1}{2}]^n$. For the central section of the cube we have the well-known result of **Ball**:

Theorem 10

For all $n \geq 2$ and all $a \in S^{n-1}$

$$A(a, 0) \leq A(\tilde{a}, 0) \quad , \quad \tilde{a} = \frac{1}{\sqrt{2}}(1, 1, 0, \dots, 0) \ .$$

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For non-central sections we have by **Moody, Stone, Zach and Zvavitch**

Theorem 11

Let $n \geq 3$, $\frac{\sqrt{n-1}}{2} < t \leq \frac{\sqrt{n}}{2}$ and $a^{(n)} = \frac{1}{\sqrt{n}}(1,\dots,1)$. Then

$$A(a,t) \leq A(a^{(n)},t) = \frac{n^{n/2}}{(n-1)!} \left(\frac{\sqrt{n}}{2} - t\right)^{n-1} \; .$$

If $n = 2$, this holds for $t > \frac{3}{8}\sqrt{2} \simeq 0.53$. It is false for $\frac{1}{2} < t < \frac{3}{8}\sqrt{2}$.

Let $\frac{\sqrt{n-1}}{2} < t \leq \frac{\sqrt{n}}{2}$ and $f := \frac{1}{2}(1, \dots, 1)$. Suppose that $\langle a, f \rangle > t$. Then $\langle a, f_i \rangle < t$ for all $f_i := (1, \dots, 1, -1, 1, \dots, 1)$, $i = 1, \dots, n$. Let $v_i = \{x \in Q_n \mid \langle a, x \rangle = t\} \cap span(f, f_i)$. Then $P := Convex(f, v_1, \dots, v_n)$ is a pyramid spanned by the vectors $v_1 - f, \dots, v_n - f$ and

$$A(a,t) = n \frac{vol_n(P)}{\frac{1}{2}\sum_{i=1}^n a_i - t}$$

since $h = \frac{1}{2}\sum_{i=1}^n a_i - t$ is the height of P.

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$$vol_n(P) = \frac{1}{n!} \det(v_1 - f, \dots, v_n - f) = \frac{1}{n!} \frac{\frac{1}{2} \sum_{i=1}^n a_i - t}{\prod_{i=1}^n a_i} \ .$$

Note that $a_i > 0$ since $0 < \langle a, f - f_i \rangle = 2a_i$. Hence under the conditions $\langle a, f \rangle > t$ and $\langle a, f_i \rangle < t$ and $a_i > 0$,

$$A(a, t) = \frac{1}{(n-1)!} \frac{(\frac{1}{2} \sum_{i=1}^n a_i - t)^{n-1}}{\prod_{i=1}^n a_i}.$$

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The diagonals provide the unique solutions in the result of Moody, Stone, Zach and Zvavitch. It has a **local** extension for slightly smaller values of t:

Theorem 12

Let $n \geq 5$ and $\frac{n-2}{2\sqrt{n}} < t \leq \frac{\sqrt{n-1}}{2}$. Then $a^{(n)}$ is at least a local maximum of $A(.,t)$.

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For $n = 3, 4$ we have a local maximum in a more restricted range of t-values closer to $\frac{\sqrt{n-1}}{2}$. The result does not hold for $n = 2$ or $n = 3$ for values of t closer to $\frac{n-2}{2\sqrt{n}}$.

Perimeter of cubic sections

An analogue of Ball's result for perimeters was shown by **Koldobsky, K.**

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Let $n \geq 3$ *and* $\tilde{a} := \frac{1}{\sqrt{2}}(1, 1, 0, \cdots, 0)$*. Then for any* $a \in S^{n-1}$

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Pełczyński had asked this question and proved it for $n = 3$ when $vol_1(\partial Q_3 \cap a^{\perp})$ is the *perimeter* of the quadrangle or hexagon of intersection.

We now consider upper estimates for the perimeter of non-central sections of the cube at distance t, $\frac{\sqrt{n-1}}{2} < t \leq \frac{\sqrt{n}}{2}$. Assume $\langle a, x \rangle = t$. On the boundary face $x_1 = \frac{1}{2}$ centered at $\frac{1}{2}(1, 0, \dots, 0)$ with $\tilde{x} = (x_2, \dots, x_n)$
$\langle \tilde{a}, \tilde{x} \rangle = \langle a, x \rangle - \frac{1}{2} a_1 = \frac{2t - a_1}{2}$, $\tilde{b} := \frac{\tilde{x}}{\sqrt{1-a_1^2}}$, $||\tilde{b}||_2 = 1$.

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Then $\langle \tilde{b}, \tilde{x} \rangle = \frac{2t - a_1}{\sqrt{1 - a_1^2}} =: s$ is the distance from the face center to $\{x \in \partial Q_n \mid x_1 = \frac{1}{2}, \langle a, x \rangle = t\}$.

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Perimeter formula for $t > \frac{1}{2}\sqrt{n-1}$

$$P(a,t) = \frac{1}{(n-2)!} \sum_{k=1}^{n} a_k \sqrt{1-a_k^2} \; \frac{(\frac{1}{2}\sum_{j=1}^n a_j - t)^{n-2}}{\prod_{j=1}^n a_j} \; .$$

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The term without the weights is maximal for $a^{(n)}$. A concavity estimate for the weights implies

Perimeter of non-central cubic sections

Theorem 14

Let $Q_n = [-\frac{1}{2}, \frac{1}{2}]^n$*,* $n \geq 4$*. Then for all* t *with* $\frac{\sqrt{n-1}}{2} < t \leq \frac{\sqrt{n}}{2}$ *and all* $a \in S^{n-1}$

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The result is also true for $n = 3$ *if* $0.725 < t \leq \frac{\sqrt{3}}{2}$.

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Again: $a^{(n)}$ is also a local maximum of $P(., t)$ if $\frac{n-2}{\sqrt{n}} < t \leq \frac{\sqrt{n-1}}{2}$, $n \geq 6$. This is true for $n = 3, 4, 5$ for some values t close to $\frac{\sqrt{n-1}}{2}$, too. E.g. for $n = 3$, $a^{(3)}$ is a local maximum of $P(., t)$ if $0.635 < t \leq \frac{\sqrt{3}}{2}$, but a local minimum if $\frac{1}{\sqrt{3}} < t < 0.635$.