Moments of random vectors

Rafał Latała (University of Warsaw)
(based on joint works with Piotr Nayar and Marta Strzelecka)

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Strong and weak moments

Let X be an n-dimensional random vector. In many problems one needs to estimate *strong moments of* X with respect to a norm structure $(\mathbb{R}^n, \|\cdot\|)$, i.e.

$$M_p(X, \|\cdot\|) := (\mathbb{E}\|X\|^p)^{1/p} = \left(\mathbb{E} \sup_{\|t\|_* \le 1} |\langle t, X\rangle|^p\right)^{1/p}, \quad p \ge 1.$$

Usually it is much easier to bound *weak moments of* X , defined as

$$\sigma_p(X, \|\cdot\|) := \sup_{\|t\|_* \le 1} (\mathbb{E}|\langle t, X\rangle|^p)^{1/p}, \quad p \ge 1.$$

It is natural to investigate relations between these quantities.

Remark. Equivalently one may take bounded nonempty subsets $T \subset \mathbb{R}^n$ and define

$$M_p(X, T) := \left(\mathbb{E} \sup_{t \in T} |\langle t, X\rangle|^p\right)^{1/p}, \quad \sigma_p(X, T) := \sup_{t \in T} (\mathbb{E}|\langle t, X\rangle|^p)^{1/p}.$$

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Question 1

Obviously weak moments are smaller than strong moments. What about the reverse inequality?
Namely, for fixed n and p what is the best constant $C_{n,p}$ such that for any random vector X and any bounded nonempty $T \subset \mathbb{R}^n$

$$(\mathbb{E} \sup_{t \in T} |\langle t, X \rangle|^p)^{1/p} \leq C_{n,p} \sup_{t \in T} (\mathbb{E} |\langle t, X \rangle|^p)^{1/p}?$$

By homogenity we may assume that weak moments are bounded by 1, i.e.

$$T \subset \mathcal{M}_p(X) := \{t \in \mathbb{R}^n \colon \ \mathbb{E}|\langle t, X \rangle|^p = 1\}$$

then for $x \in \mathbb{R}^n$

$$\sup_{t \in T} |\langle t, x \rangle| \leq \|t\|_{\mathcal{Z}_p(X)} := \sup\{|\langle t, s \rangle| \colon \ \mathbb{E}|\langle t, X \rangle|^p \leq 1\}.$$

And our goal is to find best possible $C_{n,p}$ such that

$$(\mathbb{E}\|X\|^p_{Z_p(X)})^{1/p} \leq C_{n,p}.$$

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Examples of $\mathcal{Z}_p(X)$-norms/bodies

The unit ball in norm $\|\cdot\|_{\mathcal{Z}_p(X)}$ is denoted by $\mathcal{Z}_p(X)$ and is called the L_p-centroid body of (the distribution of) X. It was introduced (under a different normalization) for uniform distributions on convex bodies by Lutvak and Zhang (1997).

- If X is isotropic then $\mathcal{Z}_2(X) = B_2^n$
- If X is the standard Gaussian then $\mathcal{Z}_p(X) \sim \sqrt{p}B_2^n$
- If X has the product symmetric exponential distribution then $\mathcal{Z}_p(X) \sim \sqrt{p}B_2^n + pB_1^n$
- If X is uniformly distributed on $\{-1,1\}^n$ or $[-1,1]^n$ then $\mathcal{Z}_p(X) \sim \sqrt{p}B_2^n \cap B_\infty^n$
- If X has a symmetric log-concave distrubution (i.e. has the density e^{-h} where $h\colon \mathbb{R}^n \to (-\infty,\infty]$ is convex) then

$$\mathcal{Z}_p(X) \sim \{t\colon \Lambda_X^*(t) \le p\},$$

where

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Rotationally invariant vectors

Consider first a vector X with rotationally invariant distribution. Then $X = RU$, where U has a uniform distribution on S^{n-1} and $R = |X|$ is a nonnegative random variable, independent of U. We have for any vector $t \in \mathbb{R}^n$ and $p \geq 2$,

$$(\mathbb{E}|\langle t, U\rangle|^p)^{1/p} = |t|(\mathbb{E}|U_1|^p)^{1/p} \sim \sqrt{\frac{p}{n+p}}|t|.$$

Therefore

$$(\mathbb{E}|\langle t, X\rangle\|^p)^{1/p} = \|R\|_{L_p}\|U_1\|_{L_p}|t| \quad \text{and} \quad \|t\|_{\mathcal{Z}_p(X)} = \|U_1\|_{L_p}^{-1}\|R\|_{L_p}^{-1}|t|.$$

So

$$\left(\mathbb{E}\|X\|_{\mathcal{Z}_p(X)}^p\right)^{1/p} = \|U_1\|_{L_p}^{-1}\|R\|_{L_p}^{-1}(\mathbb{E}|X|^p)^{1/p} = \|U_1\|_{L_p}^{-1} \sim \sqrt{\frac{n+p}{p}}.$$

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Answer to Question 1

Theorem (L-Nayar'19+)

For any n-dimensional random vector X and any nonempty set T in $\mathbb{R}^n$ and $p \geq 2$ we have

$$\left(\mathbb{E}\sup_{t\in T}|\langle t,X\rangle|^p\right)^{1/p} \leq 2\sqrt{e}\sqrt{\frac{n+p}{p}}\sup_{t\in T}\left(\mathbb{E}|\langle t,X\rangle|^p\right)^{1/p}.$$

Equivalently,

$$(\mathbb{E}\|X\|^p_{\mathcal{Z}_p(X)})^{1/p} \leq 2\sqrt{e}\sqrt{\frac{n+p}{p}}.$$

The constant is of optimal order for rotationally invariant vectors.

However for some distributions it might be smaller

Example Let $\mathbb{P}(X_i = \pm e_i) = 1/(2n)$ $i = 1,\ldots,n$ then

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Concentration for Gaussian and exponential measures

The concentration of measure phenomenon for the canonical Gaussian measure γ_n on $\mathbb{R}^n$ yields:

$$\gamma_n(A) \geq \frac{1}{2} \;\Rightarrow\; \forall_{p\geq 2}\; 1-\gamma_n(A+C\sqrt{p}B_2^n) \leq e^{-p}(1-\gamma_n(A)),$$

Talagrand's two-level concentration for the product exponential measure states that:

$$\nu^n(A) \geq \frac{1}{2} \;\Rightarrow\; \forall_{p\geq 2}\; 1-\nu^n(A+C\sqrt{p}B_2^n+CpB_1^n) \leq e^{-p}(1-\nu^n(A)).$$

Both results have the form

$$\mu(A) \geq \frac{1}{2} \;\Rightarrow\; \forall_{p\geq 2}\; 1-\mu(A+C\mathcal{Z}_p(\mu)) \leq e^{-p}(1-\mu(A)).$$

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Optimal concentration

It is not hard to show that if μ is a symmetric distribution on $\mathbb{R}^n$ (and 1-dimensional marginals of μ behave in a regular way) $p \geq p_0$ and K is a convex set such that for any halfspace H

$$1 - \mu(H + K) \leq e^{-p}$$

then $K \supset c\mathcal{Z}_p(\mu)$.

Therefore we say that a measure μ *satisfies the optimal concentration with constant C* if

$$\mu(A) \geq \frac{1}{2} \ \Rightarrow \ \forall_{p \geq 2} \ 1 - \mu(A + C\mathcal{Z}_p(\mu)) \leq e^{-p}(1 - \mu(A)).$$

Optimal concentration for log-concave vectors

All centered product log-concave measures satisfy the optimal concentration inequality with a universal constant (L-Wojtaszczyk 2008).

A natural conjecture states that this is true also for nonproduct log-concave measures. Since $\mathcal{Z}_p(X) \subset CpB_2^n$ for isotropic log-concave vectors, this is stronger than the celebrated KLS conjecture on the boundedness of the Cheeger constant for isotropic log-concave measures .

It is known that KLS holds with constant $n^{1/4}$ (Lee-Vempala), we are able to show the optimal concentration with a worse constant (but better than $\sqrt{n}$).

Corollary (L.-Nayar)

Every centered log-concave probability measure on $\mathbb{R}^n$ satisfies the optimal concentration inequality with constant $Cn^{5/12}$.

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p-summing operators

A linear operator T between Banach spaces F_1 and F_2 is p-summing if there exists a constant $\alpha < \infty$, such that

$$\forall_{x_1,\dots x_m \in F_1} \quad \left(\sum_{i=1}^{m} \| Tx_i \|^p \right)^{1/p} \le \alpha \sup_{x^* \in F_1^*, \|x^*\| \le 1} \left(\sum_{i=1}^{m} |x^*(x_i)|^p \right)^{1/p} .$$

The smallest constant α in the above inequality is called the p-summing norm of T and denoted by $\pi_p(T)$. For a Banach space F by $\pi_p(F)$ we denote the p-summing constant of the identity map of F.

It is well known that $\pi_p(F) < \infty$ if and only if F is finite dimensional. Moreover $\pi_2(F) = \sqrt{\dim F}$. Summing constants of some finite dimensional spaces were computed by Gordon. In particular he showed that

$$\pi_p(\ell_2^n) = (\mathbb{E}|U_1|^p)^{-1/p} \sim \sqrt{\frac{n+p}{p}}.$$

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p-summing constants of finite rank operators

Corollary

For any finite dimensional Banach space F and $p \geq 2$ we have

$$\pi_p(F) \leq 2\sqrt{e}\sqrt{\frac{\mathrm{dim}F + p}{p}} \leq C\pi_p(\ell_2^{\mathrm{dim}F}).$$

Proof. We apply the weak-strong comparison theorem for random vectors uniformly distributed on finite subsets of F and T the unit ball in F^*.

Corollary

Let T be a finite rank linear operator between Banach spaces F_1 and F_2. Then the p-absolutely summing constant of T satisfies

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Strong and weak moments for Gaussian vectors

Let $G = (g_1, \ldots, g_n)$, where g_i are i.i.d. $\mathcal{N}(0,1)$. Gaussian concentration states that for any L-Lipschitz function f,

$$\mathbb{P}(|f(G) - \mathbb{E}f(G)| \geq t) \leq \exp(-\frac{t^2}{2L^2})$$

Integrating by parts we get for $p \geq 1$,

$$(\mathbb{E}|f(G) - \mathbb{E}f(G)|^p)^{1/p} \leq C\sqrt{p}L.$$

Hence by the triangle inequality in L_p,

$$\|f(G)\|_{L_p} \leq |\mathbb{E}f(G)| + C\sqrt{p}L.$$

The function $x \mapsto \sup_{t\in T}|\langle t, x\rangle|$ has the Lipschitz constant $\sup_{t\in T}|t|$, moreover $\|\sum_i t_i g_i\|_{L_p} = |t|\|g_1\|_{L_p} \sim |t|\sqrt{p}$, therefore

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Strong and weak moments for Gaussian vectors

Let $G = (g_1, \ldots, g_n)$, where g_i are i.i.d. $\mathcal{N}(0,1)$. Gaussian concentration states that for any L-Lipschitz function f,

$$\mathbb{P}(|f(G) - \mathbb{E}f(G)| \geq t) \leq \exp(-\frac{t^2}{2L^2})$$

Integrating by parts we get for $p \geq 1$,

$$(\mathbb{E}|f(G) - \mathbb{E}f(G)|^p)^{1/p} \leq C\sqrt{p}L.$$

Hence by the triangle inequality in L_p,

$$\|f(G)\|_{L_p} \leq |\mathbb{E}f(G)| + C\sqrt{p}L.$$

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Question 2

What should we assume about distribution of random n-dimensional vector X in order to have for any nonempty bounded $T \subset \mathbb{R}^n$ and any $p \geq 2$,

$$\left(\mathbb{E}\sup_{t\in T}|\langle t, X\rangle|^p\right)^{1/p} \leq C_1\mathbb{E}\sup_{t\in T}|\langle t, X\rangle| + C_2\sup_{t\in T}\left(\mathbb{E}|\langle t, X\rangle|^p\right)^{1/p}.$$

with some universal constants C_1, C_2?

Rademachers and variables with log-concave tails

In the case when X_i is the Rademacher sequence (i.e. sequence of i.i.d. symmetric ± 1-valued r.v's) Dilworth and Montgomery-Smith (1993) showed that

$$\left(\mathbb{E}\sup_{t\in T}\Big|\sum_{i=1}^{n} t_i X_i\Big|^p\right)^{\frac{1}{p}} \leq C_1 \mathbb{E}\sup_{t\in T}\Big|\sum_{i=1}^{n} t_i X_i\Big| + C_2 \sup_{t\in T}\left(\mathbb{E}\Big|\sum_{i=1}^{n} t_i X_i\Big|^p\right)^{\frac{1}{p}}.$$

This inequality was generalized (L. 1996) to the case when X_i are symmetric with log-concave tails (i.e. $t \mapsto \ln \mathbb{P}(|X_i| \geq t)$ is concave from $[0,\infty)$ to $[-\infty, 0]$).

Strzelecka, Strzelecki and Tkocz (2017) showed that for symmetric variables with log-concave tails the inequality holds with $C_1 = 1$.

Estimates discussed above are strictly connected with concentration inequalities (two-level Talagrand's concetration, concentration for convex functions on discrete cube).

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Variables with sublinear growths of moments

One may show that for a r.v's X with log-concave tails $\|X\|_{L_p} \leq 2\frac{p}{q}\|X\|_{L_q}$ for $p \geq q \geq 1$.
L.-Tkocz (2015) proved that if X_i are independent, centered and

$$\|X_i\|_{L_p} \leq \alpha\frac{p}{q}\|X_i\|_{L_q} \text{ for } p \geq q \geq 1,$$

then

$$\left(\mathbb{E}\sup_{t\in T}\Big|\sum_{i=1}^n t_iX_i\Big|^p\right)^{\frac{1}{p}} \leq C_1(\alpha)\mathbb{E}\sup_{t\in T}\Big|\sum_{i=1}^n t_iX_i\Big| + C_2(\alpha)\sup_{t\in T}\left(\mathbb{E}\Big|\sum_{i=1}^n t_iX_i\Big|^p\right)^{\frac{1}{p}}.$$

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Answer to Question 2 under independence of coordinates

Theorem (L-Strzelecka'18)

Let $X_1, \ldots, X_n$ be centered, independent and

$$\|X_i\|_{L_{2p}} \leq \alpha \|X_i\|_{L_p} \qquad \text{for } p \geq 2 \text{ and } i = 1, \ldots, n, \tag{1}$$

where α is a finite positive constant. Then for $p \geq 1$ and $T \subset \mathbb{R}^n$,

$$\left(\mathbb{E} \sup_{t \in T} \Big|\sum_{i=1}^n t_i X_i\Big|^p\right)^{\frac{1}{p}} \leq C(\alpha) \left[\mathbb{E} \sup_{t \in T} \Big|\sum_{i=1}^n t_i X_i\Big| + \sup_{t \in T} \left(\mathbb{E}\Big|\sum_{i=1}^n t_i X_i\Big|^p\right)^{\frac{1}{p}}\right], \tag{2}$$

where $C(\alpha)$ is a constant depending only on α.

Remark. Symmetric r.v's such that $\mathbb{P}(|X_i| \geq t) = \exp(-t^r)$, $r \in (0,1)$ satisfy the asumptions, but do not have exponential moments, so there are no dimension-free concentration inequalities for $(X_1, \ldots, X_n)$.

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It turns out that in the i.i.d case the result may be reversed, i.e. condition (2) implies (1).

Theorem (L-Strzelecka)

Let $X_1, X_2, \ldots$ be i.i.d. random variables. Assume that there exists a constant L such that for every $p \geq 1$, every n and every non-empty set $T \subset \mathbb{R}^n$ we have

$$\Big(\mathbb{E}\sup_{t\in T}\Big|\sum_{i=1}^n t_i X_i\Big|^p\Big)^{1/p} \leq L\Big[\mathbb{E}\sup_{t\in T}\Big|\sum_{i=1}^n t_i X_i\Big| + \sup_{t\in T}\Big(\mathbb{E}\Big|\sum_{i=1}^n t_i X_i\Big|^p\Big)^{1/p}\Big].$$

Then

$$\|X_1\|_{L_{2p}} \leq \alpha(L)\|X_1\|_{L_p} \qquad \text{for } p \geq 2,$$

where $\alpha(L)$ is a constant which depends only on $L \geq 1$.

Paouris inequality

The seminal result of Paouris shows that one may compare strong and weak ℓ_2-norms of log-concave random vectors.

Theorem (Paouris 2006)

Let X be a log-concave vector. Then for $p \geq 1$,

$$(\mathbb{E}|X|^p)^{1/p} \leq C_1 \mathbb{E}|X| + C_2 \sup_{|t| \leq 1} (\mathbb{E}|\langle t, X\rangle|^p)^{1/p},$$

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Example

Let $X = gY$, where Y has uniform distribution S^{n-1} and g is $N(0,1)$ r.v. independent of Y. Then

$$(\mathbb{E}|gY|^p)^{1/p} = \|g\|_{L_p} \sim \sqrt{p},$$

and

$$\|\langle t, gY\rangle\|_{L_p} = \|g\|_{L_p}\|\langle t, Y\rangle\|_{L_p} \sim p|t|/\sqrt{n} \quad \text{for } p \leq n.$$

If we take $p = \sqrt{n}$ we see that if Paouris-type inequality holds for X:

$$(\mathbb{E}|X|^p)^{1/p} \leq C\left(\mathbb{E}|X| + \sup_{|t|\leq 1}(\mathbb{E}|\langle t, X\rangle|^p)^{1/p}\right),$$

then $\max\{C_1, C_2\} \geq cn^{1/4}$. On the other hand random variables $\langle t, X\rangle$ are very regular.

Open problem. Characterize (or at least state quite general sufficient conditions) all random vectors that satisfy the Paouris inequality.

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Paouris-type inequality for ℓ_r-norms

Theorem (L, Strzelecka 2016)

Let X be a log-concave random vector, $r < \infty$ and $\|\cdot\|$ be a norm on $\mathbb{R}^n$ such that $(\mathbb{R}^n, \|\cdot\|)$ embeds isometrically in ℓ_r. Then

$$(\mathbb{E}\|X\|^p)^{1/p} \leq Cr \left(\mathbb{E}\|X\| + \sup_{\|t\|_* \leq 1} (\mathbb{E}|\langle t, X\rangle|^p)^{1/p} \right), \qquad (3)$$

where C is a universal constant and $\|\cdot\|_*$ denotes the dual norm.

Conjecture. Inequality holds with universal constant C instead of Cr for log-concave vectors and arbitrary norm.

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Thank you for your attention!

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