An Optimal Plank Theorem

Oscar Ortega Moreno
Vienna University of Technology
Geometric Tomography

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What is a Plank?

A plank in a vector space X is the region bounded by two parallel hyperplanes.

Tarski Plank Problem

If an n-dimensional convex body is covered by a collection of planks, then the sum of the widths of the planks should be at least the minimal width of the convex body they cover.

Tarski's plank problem

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- Bang (1951): arbitrary convex bodies.

Affine Plank Problem

Bang (1951) also asked whether the widths of the planks could be measured with respect to the convex body that it is covered.

- Ball (1990) solved this affine version of the plank problem for the most interesting case: symmetric convex body.

Plank in normed spaces

A plank in a normed space X is a region of the form

$$\{x \in X : |\phi(x) - m| \leq w\}$$

where ϕ is a linear functional on X^* of norm 1, m a real number, and w is a positive number. The number w is called the half-width of the plank.

Ball's Plank theorem

Theorem (The Plank Theorem)

For any sequence $(\phi_k)_{k=1}^{\infty}$ of norm one functionals on a real Banach space X, $(m_k)_{k=1}^{\infty}$ a sequence of real numbers and non-negative numbers $(t_k)_{k=1}^{\infty}$ satisfying

$$\sum_{k=1}^{\infty} t_k < 1,$$

there exists a unit vector x in X for which

$$|\phi_j(x) - m_j| > t_j$$

for every j.

Ball's Plank theorem

The Plank Theorem is obviously sharp in the sense that the unit ball of X can be covered by n non-overlapping parallel planks whose half-widths add up to 1.

We are now going to restrict our attention to planks that are symmetric about the origin:

$$\{x \in X : |\phi(x)| \leq w\}$$

where ϕ is a linear functional on X^* of norm 1 and w is a positive number.

Theorem (The Plank Theorem)

For any sequence $(\phi_k)_{k=1}^{\infty}$ *of norm one functionals on a (real) Banach space* X *and non-negative numbers* $(t_k)_{k=1}^{\infty}$ *satisfying*

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- For other spaces we expect to be able to improve upon this condition. Hilbert Spaces?
- Ball proved that for *complex* Hilbert spaces it is possible to *beat* any sequence for which $\sum_k t_k^2 = 1$.

Complex Plank Theorem (2001)

Theorem (Complex Plank Theorem)

For any sequence $v_1, v_2, \ldots, v_n$ *of unit vectors in a complex Hilbert space* H *and positive real numbers* $t_1, t_2, \ldots, t_n$ *satisfying*

$$\sum_{k=1}^{n} t_k^2 = 1$$

there exists a unit vector $z \in H$ *such that*

$$|\langle v_k, z\rangle| \geq t_k$$

for all k*.*

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Complex Plank Theorem

Theorem (Complex Plank Theorem for same width)

For any sequence $v_1, v_2, \ldots, v_n$ of unit vectors in a complex Hilbert space H there exists a unit vector $z \in H$ such that

$$|\langle v_k, z\rangle| \geq \frac{1}{\sqrt{n}}$$

Real Hilbert spaces

What happens for *real* Hilbert spaces?

Real Hilbert spaces

This is not possible. Consider $2n$ vectors $v_1, v_2, \ldots, v_{2n}$ in $\mathbb{R}^2$ equally spaced around the circle: (n vectors and their negatives). For any unit vector v in $\mathbb{R}^2$ there is a i such that

$$|\langle v_i, v\rangle| \leq \sin(\pi/2n).$$

Fejes Tóth's zone conjecture

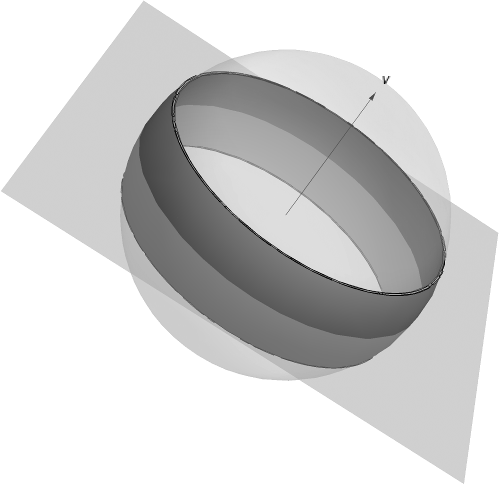
This simple statement is connected to a conjecture by Fejes Tóth that was positively answered, about two years ago, by Jiang and Polyanskii.

Zone

A zone of spherical width w associated to the great circle $S_H \cap v^\top$, for a given unit vector v in H, is the set given by

$$\{x \in S_H : |\langle v, x \rangle| \leq \sin(w/2)\}$$

Zone



In 1973, Fejes Tóth conjectured that if a collection of zones of equal width covers the unit sphere then the width of the zones should be at least π/n.

Main Theorem

Theorem (Jiang-Polyanskii 2017 ;O 2019+)

For any sequence $v_1, v_2, \ldots, v_n$ *of unit vectors in a real Hilbert space* H*, there exists a unit vector* $v \in H$ *such that*

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for all $i \in \{1, 2, \ldots, n\}$.

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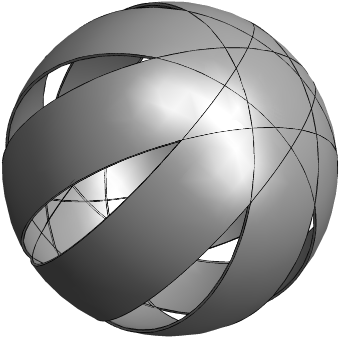
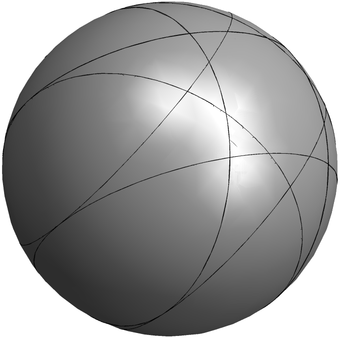
Z. Jiang and A. Polyanskii used a completely different approach.





Strategy

The basic strategy in the proof of the main theorem is the strategy followed by Ball in the proof the Complex Plank Theorem, but there is a fundamental difference.

Strategy

Fundamental difference: the main ingredient of the proof of the Complex Plank Theorem has no analogue in the real case.

Strategy

Ball studies the behaviour of a complex polynomial locally around 1 and, with the aid of the maximum modulus principle, manages to jump away from 1 to a point in the unit disk where this polynomial has large absolute value.

Strategy

In contrast, the proof of the main theorem here relies on extremal properties of trigonometric polynomials to produce this jump.

Rescaled version

Theorem

For any sequence $v_1, v_2, \ldots, v_n$ of unit vectors in a real Hilbert space H, there exists a vector $v \in H$ of norm $\sqrt{n}$ for which

$$|\langle v_k, v\rangle| \geq \sqrt{n}\sin(\pi/2n)$$

for all k.

Inverse Eigenvectors: Motivation

We want to maximize

$$\min_{1 \leq k \leq n} |\langle v_k, v \rangle|$$

subject to

$$v^\top v = n.$$

Instead...

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$$\prod_{k=1}^{n} |\langle v_k, v\rangle|$$

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and hope that the factors are large enough to get the desired inequality.

Structure of extremal points

Proposition (G. Ambrus 2009)

Let $v_1, v_2, \ldots, v_n$ *be a sequence for unit vectors in a real Hilbert space* H. *Suppose that* v *is vector of norm* $\sqrt{n}$ *chosen so as to maximize*

$$\prod_{k=1}^{n} |\langle v_k, v\rangle|.$$

Then,

$$v = \sum_{k=1}^{n} \frac{1}{\langle v_k, v\rangle} v_k$$

Structure of extremal points

Denote by H the Gram matrix $H_{ij} = \langle v_i, v_j \rangle$, and let w be the vector

$$w_k = \frac{1}{\langle v_k, v \rangle}$$

for all k. Then w satisfies

$$(Hw)_j = \sum_{i=1}^{n} h_{ji} w_i = \langle v_j, \sum_{i=1}^{n} w_i v_i \rangle = \langle v_j, v \rangle = \frac{1}{w_j}.$$

So, w satisfies the following equation $Hw = w^{-1}$ is given by

$$w^{-1} = \left(\frac{1}{w_1}, \dots, \frac{1}{w_n} \right).$$

Inverse Eigenvectors

Definition (G. Ambrus 2009)

Let M be a $n \times n$ matrix. We say that w is an *inverse eigenvector* of M if

$$Mw = w^{-1}$$

Theorem in terms of Inverse Eigenvectors

Theorem (O 2019+)

Let H be a real Gram matrix. Then, there exists an inverse eigenvector w of H for which

$$\|w\|_\infty \leq \frac{1}{\sqrt{n}\sin(\pi/2n)}$$

Final Transformation.

Lemma

Suppose that M is a symmetric positive matrix satisfying

- $M\mathbf{1} = \mathbf{1}$, *and*
- *whenever c is a vector such that*

$$c^\top M^{-1} c = n,$$

then

$$\prod |c_k| \leq 1$$

Then $m_{kk} \leq \frac{1}{n \sin^2(\pi/2n)}$ for all k.

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Then $m_{kk} \leq n$ for all k.

Proof

Let $\mathcal{E}$ be the ellipsoid defined by the equation $b^\top M b = n$, i.e.

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The proof consists of looking at a 2-dimensional "X-rays" of the ellipsoid $\mathcal{E}$ passing through the point $\mathbf{1}$. Given a vector $v \in \mathcal{E}$ orthogonal to $\mathbf{1}$, denote by H_v the subspace spanned by v and $\mathbf{1}$.

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Denote by $\mathcal{E}_v$ the 2-dimensional ellipse we get by intersecting $\mathcal{E}$ and H_v,

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Denote by $\mathcal{E}_v$ the 2-dimensional ellipse we get by intersecting $\mathcal{E}$ and H_v,

$$\begin{aligned} \mathcal{E}_v &= \mathcal{E} \cap H_v \\ &= \{\cos\theta\mathbf{1} + \sin\theta v : \theta \in [0, 2\pi]\} \end{aligned}$$

By the second condition of the lemma, for all $\theta \in [0, 2\pi]$

$$\left| \prod_{k=1}^{n} (\cos\theta + (Mv)_k \sin\theta) \right| \leq 1$$

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$$|P_v(\theta)| = \left| \prod_{k=1}^{n} (\cos\theta + (Mv)_k \sin\theta) \right| \leq 1$$

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for all $v \in \mathcal{E} \cap \mathbf{1}^\top$. In other words,

$$\|P_v\|_\infty \leq 1$$

for all $v \in \mathcal{E} \cap \mathbf{1}^\top$.

On the other hand,

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$$\frac{P_v'(\theta)}{P_v(\theta)} = -\sum_{j=1}^{n} \frac{\sin\theta - (Mv)_j \cos\theta}{\cos\theta + (Mv)_j \sin\theta}$$

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$$\frac{P_v''(\theta)P_v(\theta) - (P_v'(\theta))^2}{P_v(\theta)^2} = -\sum_{j=1}^{n} \frac{1 + (Mv)_j^2}{(\cos\theta + (Mv)_j \sin\theta)^2}.$$

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$$P'_v(0) = 0$$

$$P''_v(0) = -\sum_{j=1}^{n} 1 + (Mv)_j^2 = -(n + \|Mv\|_2^2).$$

Theorem

[Bernstein's Inequality] If P is a trigonometric polynomial of degree at most n, then

$$\left\|P'\right\|_{\infty} \leq n \left\|P\right\|_{\infty}.$$

Applying Bernstein's inequality twice, we get

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Recall that

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for all $v \in \mathcal{E} \cap \mathbf{1}^\top$.Hence,

$$n + \|Mv\|^2 = |P_v''(0)| \leq \left\|P_v''\right\|_\infty \leq n^2$$

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$$n + \left\| Mv \right\|^2 = |P_v''(0)| \leq \left\| P_v'' \right\|_\infty \leq n^2$$

for all $v \in \mathcal{E} \cap \mathbf{1}^\top$. Therefore,

$$\left\| Mv \right\|^2 \leq n(n-1)$$

for all $v \in \mathcal{E} \cap \mathbf{1}^\top$.

Now taking $v \in \mathcal{E}$ to be the eigenvector orthogonal to $\mathbf{1}$ corresponding to the largest eigenvalue λ, we get

$$n\lambda = \|Mv\|^2 \leq n(n-1)$$

and hence,

$$m_{kk} \leq \|M\|_2 = \max \lambda, 1 \leq n - 1 < n.$$

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For the optimal bound we choose a particular subspace H to bound each diagonal entry. For example, for m_{11} we pick

$$H = \{(x, y, \ldots, y) :, x, y \in \mathbb{R}\}$$